7.1. A $W^{1,p}$ function that is not in L^∞

Let $n \geq 2$ and define $u(x) = \log\left(\log\left(1 + \frac{1}{|x|}\right)\right)$. Show that $u \in W^{1,p}(B_1(0))$.

7.2. Absolute value of $u \in W^{1,p}$

Let $u \in W^{1,p}(\mathbb{R}^n)$, for $1 \leq p \leq \infty$. Prove that $|u| \in W^{1,p}(\mathbb{R}^n)$.
Hint Define

$$f_\epsilon(x) := (x^2 + \epsilon^2)^{1/2} - \epsilon, \tag{1}$$

and use the chain rule (see problem set 6) to show that $f_\epsilon \circ u \in W^{1,p}(\mathbb{R}^n)$ for all $\epsilon > 0$. Show that $f_\epsilon \circ u$ is a Cauchy sequence in $W^{1,p}(\mathbb{R}^n)$ and deduce that $|u| \in W^{1,p}(\mathbb{R}^n)$, after proving that $f_\epsilon \circ u \to u$ in $L^p(\mathbb{R}^n)$ as $\epsilon \to 0$.

7.3. Trace-zero functions in $W^{1,p}$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^1, and $1 \leq p \leq \infty$.

(a) Show that if $u \in W_0^{1,p}(\Omega)$, then $u|_{\partial\Omega} = 0$.

(b) Suppose $u \in W^{1,p}(\Omega)$ has $u|_{\partial\Omega} = 0$. Show that $u \in W_0^{1,p}(\Omega)$ by following these steps:
i) Using a partition of unity and local coordinate charts, show that it suffice to prove the following statement: if $u \in W^{1,p}(\mathbb{R}^n_+)$ has compact support in $\overline{\mathbb{R}^n_+} = \{(x', x_n) \colon x_n \geq 0\} \subset \mathbb{R}^n$ and $u|_{\mathbb{R}^{n-1}\times\{0\}} = 0$, then $u \in W_0^{1,p}(\mathbb{R}^n_+)$.
ii) By approximating u as in part (a) by a sequence of C^1 functions $u_k \in C^1(\overline{\mathbb{R}^n_+})$ and using the fundamental theorem of calculus, show that

$$\int_{\mathbb{R}^{n-1}} |u(x', x_n)|^p dx' \leq C x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^n} |Du|^p dx' dt. \tag{2}$$

iii) Let $\eta \in \mathcal{C}^\infty([0,\infty))$ be equal to 1 on $[0,1]$ and equal to 0 on $[2,\infty)$. Define $u_m(x', x_n) := (1 - \eta(m x_n))u(x', x_n)$. Show that $u_m \to u$ in $W^{1,p}(\mathbb{R}^n_+)$ as $m \to \infty$.
iv) Show that $u \in W_0^{1,p}(\mathbb{R}^n_+)$.

7.4. Simple Hölder spaces

Let $\Omega \subseteq \mathbb{R}^n$ be open and let $\alpha \in (0,1)$. Show that $C^{0,\alpha}(\bar{\Omega})$ is a Banach space.

7.5. Extra: Properties of the characteristic function

Consider $W^{1,p}(\mathbb{R}^n)$ for $n \geq 1$. Let $\mathbf{1}_{B_1(0)}$ be the characteristic function of the unit ball, i.e.

$$\mathbf{1}_{B_1(0)}(x) := \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases} \tag{3}$$

(a) Clearly, $\mathbf{1}_{B_1(0)} \in W^{1,p}(B_1(0))$. Show that $\mathbf{1}_{B_1(0)}$ cannot be approximated in$W^{1,p}(B_1(0))$ with functions $\{u_k\}_{k\in\mathbb{N}}$ in $C_c^\infty(B_1(0))$. This proves that $W_0^{1,p}(B_1(0)) \neq W^{1,p}(B_1(0))$.

(b) Prove that $\mathbf{1}_{B_1(0)}$ does not admit a weak derivative in $W^{1,p}(\mathbb{R}^n)$.

Hint. For $n = 1$ we know from the lecture that $\frac{\mathrm{d}\mathbf{1}_{B_1(0)}}{dx} = -\delta(x-1) + \delta(x+1)$. For $n \geq 2$ use Stokes' theorem.[1]

[1] As an extra challenge one could even prove the above statements for an arbitrary bounded open set $\Omega \subseteq \mathbb{R}^n$ with C^∞ boundary.