8.1. A Poincaré-like inequality on the unit ball.

Let $n \geq 1$, $\alpha > 0$, and $U = B(0,1) \subset \mathbb{R}^n$. Prove that there exists a $C > 0$ depending on only α and n such that

$$\int_U |u|^2 \,\mathrm{d}x \leq C \int_U |\nabla u|^2 \,\mathrm{d}x, \tag{1}$$

for all $u \in H^1(U)$ that satisfy

$$\lambda\left(\{u \in H^1(U) \mid u(x) = 0\}\right), \geq \alpha \tag{2}$$

where λ is the Lebesque measure.

Hint. Use Rellich compactness.

8.2. A variant of Hardy's inequality.

For $n \geq 3$ prove there exists a constant C so that for all $u \in H^1(\mathbb{R}^n)$ one has

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} \,\mathrm{d}x \leq C \int_{\mathbb{R}^n} |\nabla u|^2 \,\mathrm{d}x. \tag{3}$$

In order to prove this one might want to proceed as follows.

(a) Argue that it suffices to prove (3) for all $C_c^\infty(\mathbb{R}^n)$.

(b) Prove that for all $u \in C_c^\infty(\mathbb{R}^n)$ we have that

$$\left| Du + \lambda \frac{x}{|x|^2} u \right| \geq 0 \text{ for all } \lambda > 0. \tag{4}$$

(c) Argue that we have

$$\int \frac{x \cdot \nabla u}{|x|^2} u \,\mathrm{d}x = \int \frac{x}{2|x|^2} \cdot \nabla(u^2) \,\mathrm{d}x, \tag{5}$$

and prove the statemnt by choosing $\lambda > 0$ in **(b)** carefully.

8.3. Uniform bounds on functions in $W^{n,1}(\mathbb{R}^n)$

Let $n \geq 1$. Recall that we define $W^{n,1}(\mathbb{R}^n)$ (not $W^{1,n}(\mathbb{R}^n)$!) as

$$W^{n,1}(\mathbb{R}^n) = \left\{u \in L^1(\mathbb{R}^n \mid \partial^\alpha u \in L^1(\mathbb{R}^n) \text{ for any} \alpha \in \mathbb{N}^n, |\alpha| \leq n\right\}. \tag{6}$$

Prove that there exists a continuous embedding

$$W^{n,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n). \tag{7}$$

8.4. Horizontal derivatives.

Given $u \in H^2(\mathbb{R}^n_+) \cap H^1_0(\mathbb{R}^n_+)$ prove that $\frac{\partial u}{\partial x_i} \in H^1_0(\mathbb{R}^n_+)$, for all $i \in \{1, ..., n-1\}$.
Hint. For $u \in H^2(\mathbb{R}^n_+) \cap H^1_0(\mathbb{R}^n_+)$, consider $D_{h,i}u : \mathbb{R}^n_+ \to \mathbb{R}$ by

$$D_{h,i}u(x) = \frac{u(x + he_i) - u(x)}{h}, \tag{8}$$

with e_i the $i - th$ unit vector in $\mathbb{R}^n$.