9.1. Another density statement

(a) Let $1 \le p < n$. Show that there exists a sequence $(\psi_k)_{k\in\mathbb{N}} \subset \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $0 \le \psi_k \le 1$ so that for all k, $\psi_k(x) = 1$ for x in a neighborhood of 0, but $\psi_k \to 0$ almost everywhere and $\nabla\psi_k \to 0$ in L^p.
Hint. Try $\psi_k(x) = \psi(kx)$ for a suitable fixed function $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$.

(b) Prove the statement of part (i) also for $p = n$.

(c) Let $u \in W^{1,q}(\mathbb{R}^n)$ where $1 \le q \le n$. Show that there exists a sequence $(u_k)_{k\in\mathbb{N}} \subset W^{1,q}(\mathbb{R}^n)$ converging to u so that for all k, the function u_k equals 0 in an open neighborhood of 0.

(d) Conclude from part (iii) that $\mathcal{C}_c^\infty(\mathbb{R}^n \setminus \{0\})$ is dense in $W^{1,q}(\mathbb{R}^n)$, and therefore $W_0^{1,q}(\mathbb{R}^n) = W^{1,q}(\mathbb{R})$.

9.2. Weak solutions to the Dirichlet problem are continuous

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain.

(a) Let $g \in \mathcal{C}^\infty(\partial\Omega)$. Let $V := \{u \in H^1(\Omega) \colon u|_{\partial\Omega} = g\}$, and set $E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x$ for $u \in V$. Show that $E(u)$ has a unique minimizer $u_0 \in V$, and prove that u_0 is a weak solution of $-\Delta u_0 = 0$, $u_0|_{\partial\Omega} = g$.

(b) Continuing part (a), suppose that $|g| \le c$ on $\partial\Omega$. Show that $|u_0(x)| \le c$ for all $x \in \Omega$ as follows: set

$$F(s) := \begin{cases} -c, & s < -c, \\ s, & -c \le s \le c, \\ c, & s > c. \end{cases}$$

Show that $F \circ u_0 \in V$ and $E(F \circ u_0) \le E(u_0)$. Use the uniqueness claim of part (a) to conclude, and deduce that

$$\|u\|_{L^\infty(\Omega)} \le \|u|_{\partial\Omega}\|_{L^\infty(\partial\Omega)}$$

whenever $u \in H^1(\Omega)$ satisfies $-\Delta u = 0$.

(c) Suppose now that $u \in H^1(\Omega)$ solves $-\Delta u = 0$ and $g := u|_{\partial\Omega} \in \mathcal{C}^0(\partial\Omega)$. Show that $u \in \mathcal{C}^0(\overline{\Omega})$. Proceed as follows. Let $g_k \in \mathcal{C}^\infty(\overline{\Omega})$ be a sequence with $g_k|_{\partial\Omega} \to g$ in $\mathcal{C}^0(\partial\Omega)$ (that is, in sup norm), and let $v_k \in H_0^1(\Omega)$ be the unique weak solution of $-\Delta v_k = f_k$ where $f_k := \Delta g_k$. Show that $u_k := v_k + g_k \in \mathcal{C}^\infty(\overline{\Omega})$ satisfies $-\Delta u_k = 0$ and $u_k|_{\partial\Omega} = g_k|_{\partial\Omega}$. Conclude that $u_k - u \to 0$ in $L^\infty(\Omega)$, and use this to finish the proof.

9.3. Weak solutions to the biharmonic equation.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary.

(a) Prove that

$$\langle u, v\rangle := \int_\Omega \Delta u \Delta v dx \tag{1}$$

defines a scalar product on $H^2(\Omega) \cap H_0^1(\Omega)$ which is equivalent ot the standard scalar product $(\cdot, \cdot)_{H^2(\omega)}$.

(b) Show that $(H^2(\Omega) \cap H_0^1(\Omega), \langle\cdot,\cdot\rangle)$ is a Hilbert space.

(c) Prove that given $f \in L^2(\Omega)$ there exists a unique $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying

$$\int_\Omega \Delta u \Delta v dx = \int_\Omega f v dx \text{ for all } v \in H^2(\Omega) \cap H_0^1(\Omega). \tag{2}$$

In fact, show that $u \in \Xi := \{u \in H^4(\Omega) \cap H_0^1(\Omega) \mid \Delta u \in H_0^1(\Omega)\}$

9.4. Weak solution to a semilinear equation.

Let $u \in H^1(\mathbb{R}^n)$ be a weak solution to

$$-\Delta u + c(u) = f \text{ in } \mathbb{R}^n, \tag{3}$$

where $f \in L^2(\mathbb{R}^n)$ and $c : \mathbb{R} \to \mathbb{R}$ smooth, with $C(0) = 0$ and $c' \geq 0$. Assume as well that u has compact support, and prove that $u \in H^2(\Omega)$.

9.5. RECAP 1: Fundamental solution to Poisson's equation on $\mathbb{R}^n$

The last two exercises on this sheet are intended as a recap for deriving the fundamental solution to the Dirichlet problem on a unit ball. In this first problem, we will consider a solution to Poisson's equation on $\mathbb{R}^n$ for $n \geq 0$, given by

$$-\Delta u = f \quad \text{on} \quad \mathbb{R}^n \tag{4}$$

for $f \in C_c^0(\mathbb{R}^n)$ and $u \in C^2(\mathbb{R}^n)$. Note that when $f \equiv 0$ equation (4) is referred to as the Laplace equation.

(a) *(Radial solution to the Laplace equation on $\mathbb{R}^n \backslash \{0\}$.)* First we will attempt to find a solution to

$$\Delta u = 0 \text{ on } \mathbb{R}^n \backslash \{0\}. \tag{5}$$

As Δ is spherically symmetric, it seems reasonable to first attempt to find a spherically symmetric solution to Laplace's equation. Argue by transforming to spherical coordinates that a spherically symmetric solution

$$u(r, \theta_1, .., \theta_{n-1}) = v(r) \tag{6}$$

to the *radial equation* is given by

$$\begin{cases} v(r) = b\log(r) + c & \text{for } \ (n=2) \\ v(r) = \frac{b}{r^{n-2}} + c & \text{for } \ (n \geq 3), \end{cases} \tag{7}$$

where b, c are constants.

(b) *(Fundamental solution to Laplace's equation on $\mathbb{R}^n$.)* Inspired by the previous exercise let us set

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi}\log(|x|) & \text{for } \ (n=2) \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & \text{for } \ (n \geq 3) \end{cases}, \tag{8}$$

where $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$ and $|x| = \sqrt{|x_1|^2 + ... + |x_n|^2}$. Furthermore, given $f \in C_c^0(\mathbb{R}^n)$, define

$$u(x) := \Phi * f(x) = \begin{cases} -\frac{1}{2\pi}\int_{\mathbb{R}^2}\log(|x-y|)f(y)dy & \text{for } \quad (n=2) \\ = \frac{1}{n(n-2)\alpha(n)}\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-2}}dy & \text{for } \quad (n \geq 3). \end{cases} \tag{9}$$

Show that $u(x) := \Phi * f(x)$ solves (4) and that $u \in C^2(\mathbb{R}^n)$, whenever $f \in C_c^2(\mathbb{R}^n)$ [1]
Hint. Note that Φ is not summable near 0. Make careful estimates on a ball of radius ϵ and its complement on $\mathbb{R}^n$.

Remark. The solution Φ in equation (8) is referred to as the fundamental solution to the Laplace equation, in the sense that it solves $-\Delta\Phi = \delta_0$ in the distirubtional sense, i.e. we formally compute

$$-\Delta u(x) = \int_{\mathbb{R}^n} -\Delta_x\Phi(x-y)f(y)dy = \langle \delta_x, f\rangle = f(x). \tag{10}$$

9.6. RECAP 2: Fundamental solution to Poisson's equation on the unit ball.

In this exercise we continue with solving Poisson equation on bounded domains, specifically the unit ball.

(a) Let $\Omega \subsetneq \mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the Poisson equation

$$-\Delta u = f \ \text{ on } \ \Omega \tag{11}$$
$$u = g \ \text{ on } \ \partial\Omega, \tag{12}$$

[1] The assumptions on f in this exercise are actually too strong and regularity on f can in fact be weakened to still allow $u \in C^2(\mathbb{R}^n)$.

where $u \in C^2(\Omega)$, $f \in C^0(\Omega)$ and $g \in C^0(\partial\Omega)$. Prove using Green's formula

$$u(x) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Phi}{\partial \nu} dS(y) - \int_{\Omega} \Phi(y-x)\Delta u(y) dy, \tag{13}$$

where $\frac{\partial}{\partial \nu}$ is the directional derivative in the direction of the outward pointing unit normal at $\partial\Omega$.

Hint. For $x \in \Omega$ choose $\epsilon > 0$ such that $B(x,\epsilon) \subset \Omega$. Make careful estimates on $B(x,\epsilon)$ $V_\epsilon := \Omega \backslash B(x,\epsilon)$. Integration by parts may prove to be useful. Recall that Green's formula is given by

$$\int_{\Omega} u\Delta v - v\Delta u dx = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} dx, \tag{14}$$

for $u, v \in C^2(\bar{\Omega})$.

(b) *(Representation formula using Green's functions.)* For fixed $x \in \Omega$ we now introduce the *corrector function* ϕ^x as the solution to the boundary problem

$$\begin{cases} \Delta \phi^x = 0 & \text{on } \Omega \\ \phi^x(y) = \Phi(y-x) & \text{on } \partial\Omega. \end{cases} \tag{15}$$

We define the *Green's function* for the region Ω as

$$G(x,y) := \Phi(y-x) - \phi^x(y) \text{ for } x \neq y \in \Omega. \tag{16}$$

Prove that if $u \in C^2(\bar{\Omega})$ solves (20) then

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y) dS(y) + \int_{\Omega} f(y)G(x,y) dy \ \text{ for } \ x \in \Omega, \tag{17}$$

where $\frac{\partial G}{\partial \nu} := D_y G(x,y) \cdot \nu(y)$.

(c) Let $\Omega = B(0,1)$. Given our representation formula from **(b)** and our explicit knowledge of Φ from exercise 5, solving Poisson's equation on the unit ball only comes down to finding the corrector function $\phi^x = \phi^x(y)$ that solves

$$\begin{cases} \Delta \phi^x = 0 & \text{on } B(0,1) \\ \phi^x = \Phi(y-x) & \text{on } \partial B(0,1). \end{cases} \tag{18}$$

We proceed with this by inverting the singularity for Φ at 0 from $x \in B(0,1)$ to $\tilde{x} \in \mathbb{R}^n \backslash B(0,1)$. More specifically define the inversion $\tilde{x} = \frac{x}{||x||^2}$. Prove that ϕ^x solving (18) is given by

$$\phi^x(y) = \Phi(|x|(y - \tilde{x})). \tag{19}$$

(d) *(Poisson's integration kernel).* Putting all of the questions **(a)**-**(c)** together, we can now give an explicit formula for the Laplace equation (i.e. when $f \equiv 0$) on the unit ball, i.e. we a solution to

$$-\Delta u = 0 \quad \text{on } \Omega \tag{20}$$

$$u = g \quad \text{on } \partial\Omega, \tag{21}$$

where $u \in C^2(\Omega)$, and $g \in C^0(\partial\Omega)$. Calculate $\frac{\partial G}{\partial \nu}$ on $\partial B(0,1)$, and conclude by putting questions **(a)**-**(c)** together, that u is given by

$$u(x) = \int_{\partial B(0,1)} K(x,y) g(y) dS(y), \tag{22}$$

where *Poisson's kernel* is given by

$$K(x,y) = \frac{1 - |x|^2}{n\alpha(n)} \frac{1}{|x-y|^n}. \tag{23}$$

With this formula, prove that $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$.[2] Can you give the formula for $K(x,y)$ for a ball of arbitrary radius $r > 0$?

[2] In fact it turns out that u is smooth, another consequence of elliptic regularity!