1.1. The closure of the derivative operator.

(a) Let $A_1 = \frac{\mathrm{d}}{\mathrm{d}t}$ on $D(A_1) = \{f \in C^1([0,1]) : f'(0) = f(0) = 0 = f(1) = f'(1)\} \subset C^0([0,1])$. We claim that A_1 is the closed extension of A, i.e. $D(A_1) = D(\overline{A})$.
First of all A_1 is clearly closed: if $(f_n, f_n')_{n\in\mathbb{N}}$ is some sequence converging in $\Gamma_{A_1} \subset C^0([0,1]) \times C^0[0,1]$ to some (f,g) we have that

$$||f_n - f||_\infty \to 0, \text{ and } ||A_1 f_n - g||_\infty = ||f_n' - g||_\infty \to 0, \tag{1}$$

where $||f||_\infty = \sup_{[0,1]} |f|$ is the sup norm on $C^0([0,1])$. From this we deduce that f_n is in fact a Cauchy sequence in $C^1([0,1])$ with respect to the C^1 norm

$$||f||_{C^1} = ||f||_\infty + ||f'||_\infty.$$

As $C^1[0,1]$ is Banach we conclude that f_n converges in $C^1[0,1]$ to f and that $f' = g$. To finish the argument we remark that $f(0) = 0 = f(1)$, which follows trivially from the fact that uniform convergence implies pointwise convergence and that $f_n(0) = 0 = f_n(1)$ for all $n \in \mathbb{N}$. The same holds for the equality $f'(0) = 0 = g'(0)$.
Clearly we have $D(A) \subseteq D(A_1)$ and as A_1 is a closed extension of A we also have $D(\overline{A}) \subseteq D(A_1)$. The reverse inclusion $D(\overline{A}) \supseteq D(A_1)$ follows from the fact that every function $f \in C^1([0,1])$ which vanishes on $\{0,1\}$ can be approximated in the C^1 norm by smooth functions in $C_c^\infty((0,1))$ (again for these $f \in D(A_1)$ we have $f_n \in C_c^\infty((0,1))$ such that $||f_n - f||_\infty \to 0$ and $||f_n' - f'||_\infty \to 0$. Thus we have that $\Gamma_{A_2} \subseteq \overline{\Gamma_A} = \Gamma_{\overline{A}}$ hence $\overline{A}$ must be a closed extension of A_1 as well.

(b) We claim that $A_2 = \frac{\mathrm{d}^2}{\mathrm{d}t^2}$ on

$$D(A_2) = C^2([0,1]) \tag{2}$$

is the closed extension of $D(A)$, i.e. $D(\overline{A}) = D(A_2)$. First let $(f_n, f_n'')_{n\in\mathbb{N}}$ be a sequence in $\Gamma_{A_2} \subset C^0([0,1]) \times C^0[0,1]$ converging to some (f,g). We then have that

$$||f_n - f||_\infty \to 0, \text{ and } ||A_2 f_n - g||_\infty = ||f_n'' - g||_\infty \to 0. \tag{3}$$

We define the alternative norm on $C^2([0,1])$ as

$$||f||_{C^2,a} = ||f||_\infty + ||f''||_\infty. \tag{4}$$

Certainly with respect to this norm the sequence f_n would be Cauchy. We show that this norm is in fact equivalent to the standard C^2 norm hence making $(C^2([0,1]), ||\cdot||_{C^2_a})$ a Banach space. Clearly we have

$$||f||_{C^2,a} \leq ||f||_{C^2} := ||f||_\infty + ||f'||_\infty + ||f''||_\infty. \tag{5}$$

Thus we only look for a $c > 0$ such that

$$||f||_{C^2} \leq c||f||_{C^2,a}. \tag{6}$$

For $1 \geq t \geq \frac{1}{2}$ we Taylor expand

$$f(0) = f(t) - tf'(t) + \frac{t^2}{2}f''(\xi) \tag{7}$$

for some $\xi \in [\frac{1}{2}, 1)$. We rearrange this to

$$|f'(t)| \leq |t|^{-1}(|f(0)| + |f(t)|) + \frac{|t|}{2}|f''(\xi)| \leq 4\sup_{[0,1]}|f| + \frac{1}{2}\sup_{[0,1]}|f''| \leq 4||f||_{C^2,a}. \tag{8}$$

Similarly for $0 \leq t < \frac{1}{2}$, we write

$$f(1) = f(t) + (1-t)f'(t) + \frac{(1-t)^2}{2}f''(\xi), \tag{9}$$

for some $\xi \in [0, 1/2)$. We rearrange this again to

$$f(1) = f(t) + (1-t)f'(t) + \frac{(1-t)^2}{2}f''(\xi), \tag{10}$$

from which we again deduce

$$|f'(t)| \leq |t|^{-1}(|f(0)| + |f(t)|) + \frac{|t|}{2}|f''(\xi)| \leq 4\sup_{[0,1]}|f| + \frac{1}{2}\sup_{[0,1]}|f''| \leq 4||f||_{C^2,a}. \tag{11}$$

Thus given that the norms are equivalent and $C^2[0,1]$ is Banach with respect to the standard C^2 norm, we know indeed that the sequence f_n in (3) indeed converges in C^2 norm to f. This shows us that A_2 is closed and a closed extension of A. To show that $D(A_2) \subseteq D(\overline{A})$ we need to show that $\Gamma_{A_2} \subseteq \overline{\Gamma_A} = \Gamma_{\overline{A}}$. In this case this is equivalent to showing that any function in $C^2[0,1]$ can be approximated in C^2 norm by functions in $C^\infty([0,1])$. This follows from an even stronger statement: the polynomials are dense in $C^k([0,1])$ for each $k \in \mathbb{N}_0$, which we will prove for completeness' sake. For $k = 0$ this is the famous Stone-Weierstrass theorem. Assume the statement is true for $k = n-1$. Let $f \in C^k([0,1])$; then we can write

$$f(t) = f(0) + \int_0^t f'(x)dx.$$

Then as $f' \in C^k([0,1])$ we know there exists a sequence of polynomials p_n such that $||f' - p_n||_{C^{k-1}} \to 0$. Defining the polynomial $q_n(t) := f(0) + \int_0^t p_n'(x)dx$ it is now easy to see that $||f - q_n||_{C^k} \to 0$ as we are only left to check

$$\begin{aligned} ||f - q_n||_\infty &= \sup_{t\in[0,1]} \left| \int_0^t f'(x) - p_n(x) dx \right| \\ &\leq \sup_{t\in[0,1]} \int_0^t |f'(x) - p_n(x)| dx \\ &\leq 1 \cdot \sup_{x\in[0,1]} |f'(x) - p_n(x)| \\ &= ||f' - p_n||_\infty \to 0. \end{aligned}$$

(c) To check whether the operator is closable we want use lemma $L.2$. That is, we consider a sequence of functions $u_n \in D(A) := C^1([0,1])$ such that $||u_n||_{L^2} \to 0$ and aim to show that then also $||Au_n||_{L^2} \to 0$. Let us set $v_n := \frac{\mathrm{d}}{\mathrm{dt}} u_n$ and suppose $v \in L^2([0,1])$ is a limit of the v_n, i.e. $||v - v_n||_{L^2} \to 0$. We will show $v = 0$. Using Hölder's inequality, we have for arbitrary $\phi \in C_c^\infty((0,1))$ that

$$\left| \int_0^1 v_n(t)\phi(t) dt \right| = \left| - \int_0^1 u_n(t)\phi'(t) dt \right| \leq ||u_n||_{L^2} ||\phi||_{L^2} \to 0 \text{ as } n \to \infty, \tag{12}$$

where we used integration by parts in the second step. On the other hand we know (e.g. by continuity of the L^2-scalar product) that

$$\int_0^1 v(t)\phi(t) dt = \lim_{n\to\infty} \int_0^1 v_n(t)\phi(t) dt. \tag{13}$$

and since $\phi \in C_c^\infty((0,1))$ was arbitrary we conclude that $v = 0$ by theorem $T.2$.

1.2. An operator that is *not* closable

We remark first that this exercise is a slightly more general stated version of the example $E.4$ *iii*) stated in the lectures (where $f \equiv 1 \in L^\infty(\mathbb{R}) \setminus L^2([0,1])$). Let us prove this special case first before moving on to the general case. We want to use lemma $L.2$ again. We want to show that there exists a sequence $(u_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} ||u_n||_{L^2} = 0$ but $\lim_{n\to\infty} |Au_n| \neq 0$. Let us define

$$u_n = \frac{1}{n} \mathbf{1}_{[0,n]} \tag{14}$$

Clearly we have

$$||u_n||_{L^2} = \frac{\sqrt{n}}{n} \to 0 \text{ as } n \to \infty, \tag{15}$$

however on the other hand we have

$$|Au_n| = \left| \lim_{n\to\infty} \int_{-\infty}^\infty \frac{1}{n} \mathbf{1}_{[0,n]}(x) dx \right| = \frac{n}{n} = 1. \tag{16}$$

so Au_n does not converge to 0 in $\mathbb{C}$.
Inspired this example we now prove the more general statement in the exercise. Let $f \in L^\infty(\mathbb{R}) \setminus L^2(\mathbb{R})$. We then know that

$$\int_{-\infty}^{\infty} |f|^2 dx = \infty. \tag{17}$$

Hence for each n there must exist an $x_n \in \mathbb{R}$ such that

$$\int_{-x_n}^{x_n} |f|^2 dx = n. \tag{18}$$

We then modify our previous set of u_n and define

$$u_n = \frac{1}{n} f \mathbf{1}_{[-x_n, x_n]}. \tag{19}$$

We see again that

$$\begin{aligned} ||u_n||_{L^2} &= \left(\frac{1}{n^2} \int_{-\infty}^{\infty} |f|^2 \mathbf{1}_{[-x_n, x_n]} dx \right)^{\frac{1}{2}} \\ &= \frac{1}{n} \left(\int_{-x_n}^{x_n} |f|^2 dx \right)^{\frac{1}{2}} \\ &= \frac{\sqrt{n}}{n} \to 0 \text{ as } n \to \infty, \end{aligned}$$

but that

$$\begin{aligned} |Au_n| &= \frac{1}{n} \int_{-\infty}^{\infty} |f|^2 \mathbf{1}_{[-x_n, x_n]} dx \\ &= \frac{1}{n} \int_{-x_n}^{x_n} |f|^2 dx \\ &= \frac{n}{n} = 1 \end{aligned}$$

which does not converge to 0 in $\mathbb{C}$.

1.3. Closed Sum

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $D(A)$. Then by the triangle inequality and the assumption we have

$$\begin{aligned} ||A(x_n - x_m)||_Y - ||(A+B)(x_n - x_m)||_Y &\leq ||B(x_n - x_m)||_Y \\ &\leq a||A(x_n - x_m)||_Y + b||x_n - x_m||_X, \end{aligned}$$

which implies the estimate given in the hint:

$$(1-a)||A(x_n - x_m)||_Y \leq ||(A+B)(x_n - x_m)||_Y + b||x_n - x_m||_X. \tag{20}$$

Assume that $x_n \to x$ in X and $(A+B)x_n \to y$ in Y. The claim is $(A+B)x = y$. Since $a < 1$ the estimate in (20) implies that $(Ax_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $(Y, ||\cdot||_Y)$ and therefore converges to some $\tilde{y}$. Since the graph of A is closed by assumption, we have $x \in D_A$ with $Ax = \tilde{y}$. Therefore, we may conclude

$$||B(x - x_n)||_Y \leq a||A(x - x_n)||_Y + b||x - x_n|| \xrightarrow{n\to\infty} 0, \tag{21}$$

which implies $Bx_n \to Bx$ in Y and thus

$$y = \lim_{n\to\infty}(A+B)x_n = \lim_{n\to\infty} Ax_n + \lim_{n\to\infty} Bx_n = Ax + Bx = (A+B)x. \tag{22}$$

1.4. Closable Inverse

Since the closure $\bar{A}$ is assumed to be injective, A is injective and therefore has inverse $A^{-1} : W_A \to D_A$, where $W_A := A(D_A)$ denotes the range of A. Defining

$$\begin{aligned} \chi : X \times Y &\to Y \times X \\ (x, y) &\mapsto (y, x), \end{aligned}$$

we observe that the graph $\Gamma_{A^{-1}}$ of A^{-1} is given by

$$\Gamma_{A^{-1}} := \Big\{(y, x) \in Y \times X \;:\; y \in W_A, x = A^{-1}y\Big\} = \chi(\Gamma_A). \tag{23}$$

Since χ is an isomorphism of normed spaces, we have

$$\overline{\Gamma_{A^{-1}}} = \overline{\chi(\Gamma_A)} = \chi(\Gamma_{\bar{A}}) = \Gamma_{(\overline{A})^{-1}}. \tag{24}$$

Since this proves that $\overline{\Gamma_{A^{-1}}}$ is the graph of the linear operator $(\bar{A})^{-1}$ (which is well-defined, since $\overline{A}$ is injective). Therefore, A^{-1} is closeable as claimed and

$$\Gamma_{\overline{A^{-1}}} = \overline{\Gamma_{A^{-1}}} = \Gamma_{(\bar{A})^{-1}} \implies \overline{A^{-1}} = (\bar{A})^{-1}. \tag{25}$$