10.1. Sufficient conditions for a solutions to an elliptic PDE.

1.) Let us note that first case is implied by the second case, so we immediately move over to that one.

2.) Let us formulate the elliptic PDE in the weak sense. For $u \in H_0^1(\Omega)$ to solve

$$Lu(x) = -\sum_{i,j=1}^{n} \partial_i(g^{ij}(x)\partial_j u(x)) + \sum_{i=1}^{n} b^i(x)\partial_i u(x) + c(x)u(x) = f(x)u(x) \quad (1)$$

weakly for $f \in L^2(\bar{\Omega})$ is equivalent to requiring that

$$\sum_{i,j=1}^{n} \int_\Omega g^{ij}\partial_i u(x)\partial_j v(x) + b^i(x)v(x)\partial_i u(x) + c(x)u(x)v(x)\, dx = \int_\Omega f(x)v(x)\, dx, \quad (2)$$

for $v \in H_0^1(\Omega)$. Let us define the bilinear map $\langle \cdot, \cdot \rangle : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$ by

$$\langle u, v\rangle_\Lambda = \sum_{i,j=1}^{n} \int_\Omega g^{ij}\partial_i u(x)\partial_j v(x) + b^i(x)v(x)\partial_i u(x) + b^i(x)u(x)\partial_i v(x) + c(x)u(x)v(x) dx, \quad (3)$$

the aim is to prove that $\langle u, u\rangle_\Lambda$ defines a norm on $H_0^1(\Omega)$ and then argue by Riesz that $\langle u, v\rangle_\Lambda = \int \tilde{f} v(x) dx$ where

$$\tilde{f}(x) := f(x) - \partial_i b^i(x)u(x) - b^i(x)\partial_i u(x).$$

Note that we need to solve for $\tilde{f}$ instead of f as we needed to add the term $b^i(x)u(x)\partial_i v(x)$ on both sides of (3) and integrate by parts on the right side to get an equivalent formulation to (2). We claim that the associated energy functional given by

$$E(u) := \sum_{i,j=1}^{n} \int_\Omega g^{ij}\partial_i u(x)\partial_j u(x) + 2b^i(x)u(x)\partial_i u(x) + c(x)u(x)^2\, dx \quad (4)$$

is positive definite provided we choose $||b||_{L^\infty} < \epsilon$ is small enough. Indeed, we can estimate as follows

$$\begin{aligned} E(u) = \langle u, u\rangle_\Lambda &= \sum_{i,j=1}^{n} \int_\Omega g^{ij}\partial_i u(x)\partial_j u(x) + 2b^i(x)u(x)\partial_i u(x) + c(x)u(x)^2\, dx \\ &\geq \lambda||\nabla u||_{L^2}^2 + \underbrace{\int_\Omega cu^2\, dx}_{\geq 0} - 2||b||_{L^\infty}\int_\Omega u\partial_i u\, dx \\ &\geq \lambda||\nabla u||_{L^2}^2 - 2||b||_{L^\infty}(||u||_{L^2}||\nabla u||_{L^2}) \\ &\geq C(\lambda - 2||b||_{L^\infty})||u||_{H_0^1}, \end{aligned}$$

where we use as in previous exercises that the $||\nabla u||_{L^2}$ and stand $||u||_{H_0^1}$ norms are equivalent. We see that for $||b||_{L^\infty} < \lambda$ we have that $E(u)$ is positive definite. Moreover the fact that $\langle u, u\rangle \geq 0$ for all $u \in H_0^1(\Omega)$ implies that it defines a norm on $H_0^1(\Omega)$ as well. In fact we claim that it equivalent to the H_0^1-norm. To this end let us compute

$$\begin{aligned} E(u) = \langle u, u\rangle_\Lambda &= \sum_{i,j=1}^n \int_\Omega g^{ij}\partial_i u(x)\partial_j u(x) + 2b^i(x)u(x)\partial_i u(x) + c(x)u(x)^2 \, dx \\ &\leq ||g^{ij}||_{L^\infty}||D^2 u||_{L^2} + 2||b^i||_{L^\infty}||u||_{L^2}||\nabla u||_{L^2} + ||c||_{L^\infty}||u||_{L^2}^2 \\ &\leq C(||g^{ij}||_{L^\infty} + 2||b^i||_{L^\infty} + ||c||_{L^\infty})||u||_{H_0^1} \end{aligned}$$

where we used that g^{ij}, b^i, c are simply attain a maximum on $\bar{\Omega}$ as they are smooth. With the above considerations we conclude that there are a $C_1, C_2 \geq 0$ such that

$$C_1||u||_{H_0^1}^2 \leq \langle u, u\rangle_\Lambda \leq C_2||u||_{H_0^1}^2.$$

so $||u||_\Lambda = \sqrt{\langle u, u\rangle_\Lambda}$ defines a norm equivalent to the $||\cdot||_{H_0^1}$ norm on $H_0^1(\Omega)$. Most importantly $(H_0^1(\Omega), ||\cdot||_\Lambda)$ is a Hilbert space. Let us define the functional $\ell_{\tilde{f}} : H_0^1(\Omega) \to \mathbb{C}$ via

$$\ell_{\tilde{f}}(v) = \int_\Omega \tilde{f}(x)v(x) \, dx. \tag{5}$$

Then by Riesz there exists a unique $u \in H_0^1(\Omega)$ such that

$$\langle u, v\rangle_\Lambda = \ell_{\tilde{f}}(v).$$

However, this is *exactly* the weak formulation (2).

3.) Following the hint we see that

$$\sum_{i,j=1}^n -\frac{1}{\gamma}\partial_i(g^{ij}\gamma\partial_j u) = -\sum_{i,j=1}^n \partial_i(g^{ij}\partial_j u) - \sum_{i,j=1}^n \underbrace{g^{ij}(x)\frac{\partial_j\gamma(x)}{\gamma(x)}}_{=b^i(x)}. \tag{6}$$

Therefore we see that that we can rewrite

$$-\sum_{i,j=1}^n \partial_i(g^{ij}(x)\partial_j u(x)) + \sum_{i=1}^n b^i(x)\partial_i u(x) = \sum_{i,j=1}^n -\frac{1}{\gamma}\partial_i(g^{ij}\gamma\partial_j u)$$

Now as $\gamma > 0$ we also know that $\min_\Omega \gamma > 0$. Therefore when we absorb γ into g^{ij} by defining

$$\tilde{g}^{ij} = \gamma g^{ij},$$

we see that the ellipticity condition is still fullfilled, i.e.

$$\sum_{i,j=1}^{n} \tilde{g}^{ij}\xi_i\xi_j \geq \lambda'$$

for some $\lambda' > 0$. Playing the same game as before we now want to weakly solve the equation

$$\int_\Omega \tilde{g}^{ij}\partial_i u\,\partial_j v\;dx = \int_\Omega f\gamma\,u\,v\;dx. \tag{7}$$

We can then again define an equivalent norm on $H_0^1(\Omega)$ by setting

$$\langle u, v\rangle_{\Lambda'} = \int_\Omega \tilde{g}^{ij}\partial_i u\,\partial_j v\;dx$$

which then again yields the existence for a weak solution with Riesz due to the ellipticity condition on $\tilde{g}^{ij}$

10.2. Energy functional for non-linear Poisson equation with cubic term.

(a) Let us break up this functional in its relevant parts. We know of course that

$$u \mapsto \int_\Omega |\nabla u|^2\;dx$$

is continuous in the H_0^1 norm as $||\nabla u||_{L^2}^2$ is equivalent to the standard norm on $H_0^1(\Omega)$. Apart from that the functional

$$u \mapsto \int_\Omega fu\;dx$$

is linear, can be majorized by $||u||_{L^2}$ norm by Cauchy-Schwartz, which in turn is a priori majorized by $||u||_{H_0^1}$. For the remaining term $F : H_0^1(\Omega) \to \mathbb{R}$ given by

$$F(u) = \int_\Omega u^4\;dx,$$

Obviously we "recognize" that this quantity would be equal to $||u||_{L^4}^4$ **if** u were to lie in $L^4(\Omega)$ as well which is not a priori clear. This is where the Sobolev embedding comes into play. [1] We know that for $n = 3$ and $p = 2$ $H_0^1(\Omega)$ embeds compactly into $L^q(\Omega)$ where for q we must have

$$q < \frac{np}{n-p} = \frac{3\cdot 2}{3-2} = 6,$$

[1] The Sobolev embeddings are always your first line of attack to tackle cubic and higher order terms in non-linear functionals.

where we emphasize that an embedding for $q = 6$ is still possible but is no longer compact. For $q = 4$ it is all fine however, so let

$$\iota : H_0^1(\Omega) \hookrightarrow L^4(\Omega) \tag{8}$$

be the compact embedding and $\tilde{F} : L^4(\Omega) \to \mathbb{R}$ be given by

$$\tilde{F}(u) = ||u||_{L^4}^4.$$

Clearly $\tilde{F}$ is continuous with respect to the norm of $L^4(\Omega)$. We then conclude that $F = \tilde{F} \circ \iota : H_0^1(\Omega) \hookrightarrow L^4(\Omega) \to \mathbb{R}$ is continuous as a composition of continuous functions.

(b) Coercivity follows very easily: note the following

$$\begin{aligned} E(u) &= \int_\Omega \frac{1}{2}||\nabla u||^2 + \underbrace{\frac{1}{4}u^4}_{\geq 0} - fu \, dx \\ &\geq \frac{1}{2}||u||_{H_0^1}^2 - ||f||_{L^2}||u||_{L^2} \\ &\geq \frac{1}{2}||u||_{H_0^1}^2 - C||f||_{L^2}||u||_{H_0^1} \\ &= \left(\frac{1}{2}||u||_{H_0^1} - C||f||_{L^2}\right)||u||_{H_0^1} \to \infty \text{ as } ||u||_{H_0^1} \to \infty. \end{aligned}$$

For weakly lower semi-continuity we break the functional up again in its relevant parts. Let $u_k \rightharpoonup u$ in $H_0^1(\Omega)$. Note that from chapter 4 in FA I we know that then

$$||u||_{H_0^1} \leq \liminf_{n\to\infty} ||u_k||_{H_0^1}$$

Therefore it is immediate that

$$\int_\Omega |\nabla u|^2 \, dx \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^2 \, dx.$$

For the quadratic term we note that the embedding $\iota : H_0^1(\Omega) \to L^4(\Omega)$ allows us to write any bounded functional $\ell \in L^4(\Omega)^*$ as a bounded functional $\tilde{\ell} := \ell \circ \iota \in H_0^1(\Omega)^*$ as

$$\ell(v) \leq C||v||_{L^4} \leq CC'||v||_{H_0^1(\Omega)}. \tag{9}$$

Thus if $u_k \to u$ in $H_0^1(\Omega)$ (meaning $\tilde{\ell}(u_k) \to \tilde{\ell}(u)$) we must also have $\ell(u_k) \to \ell(u)$ in L^4 by continuity of ι. We conclude $u_k \rightharpoonup u$ in $L^4(\Omega)$ as well, whence

$$\int_\Omega u^4 \, dx \leq \liminf_{k\to\infty} \int_\Omega u_k^4 \, dx.$$

The existence of a unique minimizer is now given by the variational principle (see chapter 7, FA I) as $H_0^1(\Omega)$ is Hilbert thus reflexive, and moreover a priori weakly sequentially closed in itself. As a recap, we do this case explicitly. We will prove along the following lines: as E is coercive, then there is some ball $B \subseteq H_0^1(\Omega)$ such that

$$\inf_{u \in H_0^1(\Omega)} E(u) = \inf_{u \in B} E(u).$$

That the infimum is not $-\infty$ follows from the fact that

$$E(u) \geq \int_\Omega fu \; dx \leq -||u||_{H_0^1}||f||_{H_0^1}$$

which is bounded from below for $u \in B \subset H_0^1(\Omega)$. Thus the infimum

$$E_- := \inf_{u \in H_0^1(\Omega)} E(u) \tag{10}$$

exists. To show that it is attained (i.e. there exists a $u \in H_0^1(\Omega)$ such that $E(u) = E_-$, let $(u_k)_{k \in \mathbb{N}} \subset B$ be the sequence such that

$$\lim_{k \to \infty} E(u_k) = \inf_B E(u) = E_-. \tag{11}$$

Then by Banach-Alaoglu, as u_k is bounded, and $H_0^1(\Omega)$ is reflexive, there exists a weakly convergent subsequence u_{k_j} such that $u_{k_j} \rightharpoonup u$ in $H_0^1(\Omega)$. Then by w.s.l.s.c. we have

$$E(u) \leq \liminf_{j \to \infty} E(u_{k_j}) = E_-.$$

But as E_- is the infimum attained on $H_0^1(\Omega)$, we must have $E(u) = E_-$.

(c) Let us assume that $v \in H_0^1(\Omega)$ is another minimizer, and consider $w = \frac{1}{2}(u+v)$. We analyze the functional again on its relevant components. First of all let us set $I : H_0^1(\Omega) \to \mathbb{R}$ as

$$I(\phi) = \int_\Omega \frac{1}{2}|\nabla \phi|^2 - fu \; dx, \quad \text{for } \phi \in H_0^1(\Omega)$$

and $J : H_0^1(\Omega) \to \mathbb{R}$ as

$$J(\phi) = \frac{1}{4}\int_\Omega \phi^4 \; dx, \quad \text{for } \phi \in H_0^1(\Omega)$$

so that $E(\phi) = I(\phi) + J(\phi)$. Let us consider I first. Note that for $w = u + v$ we have

$$\begin{aligned} I[w] &= \int_\Omega \frac{1}{2}\left|\frac{\nabla u + \nabla v}{2}\right|^2 - f \cdot \left(\frac{u+v}{2}\right) \; dx \\ &= \int_\Omega \frac{1}{8}(|\nabla u|^2 + 2\nabla u \cdot \nabla v + |\nabla v|^2) - f \cdot \left(\frac{u+v}{2}\right) \; dx. \end{aligned}$$

Now we note that

$$2\nabla u \cdot \nabla v = |\nabla u|^2 + |\nabla v|^2 - |\nabla u - \nabla v|^2.$$

Thus we get

$$\begin{aligned} I[w] &= \int_\Omega \frac{1}{8}(2|\nabla u|^2 + 2|\nabla v|^2 - |\nabla u - \nabla v|^2) - f \cdot \left(\frac{u+v}{2}\right)\, dx \\ &< \frac{1}{2}\int_\Omega \frac{1}{2}(2|\nabla u|^2 + 2|\nabla v|^2 - |\nabla u - \nabla v|^2) - f \cdot \left(\frac{u+v}{2}\right)\, dx \\ &= \frac{1}{2}\int_\Omega \frac{1}{2}(|\nabla u|^2 - fu)\, dx + \frac{1}{2}\int_\Omega \frac{1}{2}(|\nabla v|^2 - fv)\, dx \\ &= \frac{1}{2}I[u] + \frac{1}{2}I[v] \end{aligned}$$

where the strict inequality holds as we assumed $u \neq v$. Now for the J term, we note that the function $x \mapsto x^4$ is a convex function. So in particular, we have that

$$(\frac{1}{2}x + \frac{1}{2}y)^4 \leq \frac{1}{2}x^4 + \frac{1}{2}y^4. \tag{12}$$

From this it immedaitely follows that

$$\begin{aligned} J[\frac{u+v}{2}] &= \frac{1}{4}\int_\Omega \left(\frac{u+v}{2}\right)^4\, dx \\ &\leq \frac{1}{4}\int_\Omega \frac{1}{2}u^4 + \frac{1}{2}v^4\, dx \\ &\leq \frac{1}{2}J[u] + \frac{1}{2}J[v]. \end{aligned}$$

We conclude that

$$E\left[\frac{u+v}{2}\right] < \frac{1}{2}E[u] + \frac{1}{2}E[v] \tag{13}$$

which is a contradiction to the fact that $u, v \in H_0^1(\Omega)$ were assumed to be minimizers of E.

(d) The weak formulation of the PDE is given by

$$\int_\Omega \nabla u \cdot \nabla v + u^3 v\, dx = \int_\Omega fv\, dx$$

for $v \in H_0^1(\Omega)$. Now if u is the minimizer of E let us us set vary for $\phi \in H_0^1(\Omega)$ and $\epsilon > 0$ small we have that

$$0 \leq E(u + \epsilon\phi) - E(u) = \epsilon \int_\Omega \nabla u \cdot \nabla \phi + u^3\phi - f\phi\, dx + \mathcal{O}(\epsilon^2),$$

where we collect only the terms first order in ϵ. As we are free to choose $\phi \in H_0^1(\Omega)$ arbitrarily, this inequality can only hold if and only if for all $\epsilon > 0$ we have

$$\int_\Omega \nabla u \cdot \nabla \phi + u^3 \phi - f\phi \, dx = 0.$$

for all $\phi \in H_0^1(\Omega)$, i.e. if u solves

$$\nabla u + u^3 = f$$

weakly.

(e) From the way the question is formulated it is clear that we have to apply some bootstrap argument. As mentioned earlier, at the boundary case $q = 6$ we still have the Sobolev embedding

$$H_0^1(\Omega) \hookrightarrow L^6(\Omega). \tag{14}$$

Thus if $u \in H_0^1(\Omega)$ we know that $u \in L^6(\Omega)$ and therefore that $u^3 \in L^2(\Omega)$. Thus if $u \in H_0^1(\Omega)$ solves

$$-\Delta u + u^3 = f$$

weakly we also know that u solves

$$-\Delta u = g$$

where $g \in L^2(\Omega)$ is defined as $g = f - u^3$. From elliptic regularity we then see that $u \in H^2(\Omega)$. Using the Sobolev embedding again for $q = 6$ we see that we also have

$$H^2(\Omega) \hookrightarrow W^{1,6}(\Omega). \tag{15}$$

but then that means that if $u \in H^2(\Omega)$ and thus $u \in W^{1,6}(\Omega)$ then again $u^3 \in W^{1,2}(\Omega) = H^1(\Omega)$. Therefore $g := f - u^3 \in H^1(\Omega)$ which then implies in turn that $u \in H^3(\Omega)$ by higher regularity. Continuing on we see that

$$u \in \bigcap_{k=0}^{\infty} H^k(\Omega) = C^\infty(\Omega).$$

10.3. Rellich compactness for general domains.

(a) Let us define this operator first by density. Let $Q =]-L, L[^n$. Then we first define our operator on $C_c^\infty(\Omega)$. We define $E : C_c^\infty(\Omega) \to C_c^\infty(Q)$ by extending by 0

$$E(\phi)(x) = \begin{cases} \phi(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega \backslash Q. \end{cases}$$

Clearly $E(\phi)$ is still smooth on the domain $\Omega \backslash Q$. Note also that

$$||E(\phi)||_{L^2(Q)} = ||\phi||_{L^2(\Omega)} \text{ and } ||E(\nabla\phi)||_{L^2(Q)} = ||\nabla\phi||_{L^2(\Omega)}$$

so

$$||E(\phi)||_{H^1(Q)} = ||\phi||_{H^1(\Omega)}.$$

We conclude that E is an isometry and continuous. As $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ we can extend E to a map $E : H_0^1(\Omega) \to H_0^1(Q)$. To see that E is truly a well defined extension we want to show that

$$||E(u)|_\Omega - u||_{H_0^1(\Omega)} = 0 \text{ for all } u \in H_0^1(\Omega). \tag{16}$$

This follows as when we take $C_c^\infty(\Omega) \ni u_k \to u$ in $H_0^1(\Omega)$ we have that $E(u_k)|_\Omega = u_k \to E(u)|_\Omega$ in $H_0^1(\Omega)$.

(b) The problem with this question is that the domain $Q =]-L, L[^n$ does not have a C^1-boundary so we cannot instantly apply some Rellich compactness (corollary $C.8$) or Sobolev embedding result. With the particular instance of the cube Q we can circumvent this somewhat though. Let B be a ball that contains Q. Then as the hint suggests we keep reflecting across the boundaries of Q to get a larger square Q' such that B lies in Q' proper. As B is bounded we can do this in a finite number of reflections across the edges, say N times, where we reflect in every dimension fully so that the projection of B into that dimension is fully contained in Q' (i.e. for $\mathbb{R}^2$ we first reflect Q fully horizontally and then vertically so as not to get a reflection from two cubes Q_1, Q_2 on a third shared boundary cube Q_3).

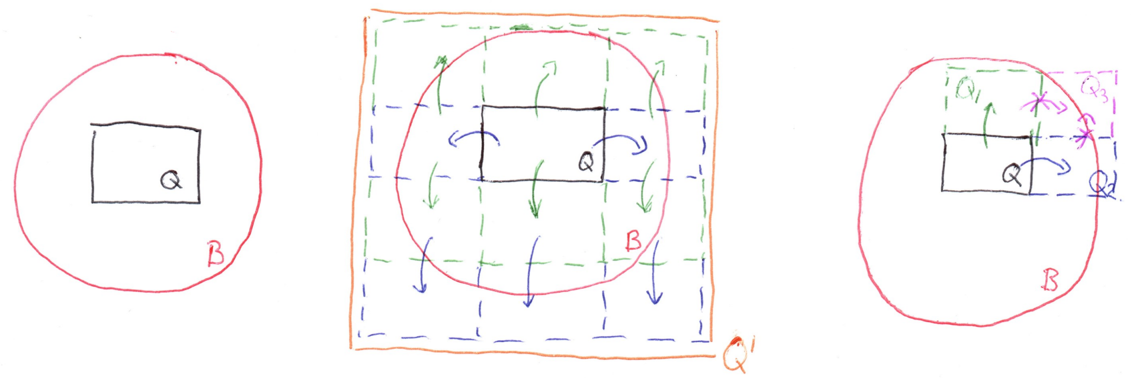

Figure 1: Left: the n-dimensional cube (black) inside the ball B (red), Center: The correct extension by reflections of Q to Q' (yellow), Right: An inconsistent extension of the cube Q.

Now using a reflection operator as defined in the lecture (e.g. as in lemma $L.14$) we also get a reflection operator $E : H^1(Q) \to H^1(Q')$ where

$$||Eu||_{H^1(Q')} \leq C||u||_{H^1(Q)}$$

for all $u \in H^1(Q')$ for some constant $C > 0$ and with $Eu|_Q = u$. Thus for $u \in H^1(Q)$ arbitrary, we have that $Eu \in H^1(Q')$ but then by restriction also $Eu \in H^1(B)$. But then $Eu \in H^1(B)$ by restriction, as $B \subsetneq Q'$. For B we can use Rellich's compactness theorem, i.e. that

$$H^1(B) \hookrightarrow L^2(B) \tag{17}$$

compactly via an embedding $\iota : H^1(B) \hookrightarrow L^2(B)$. Then restricting with a map $r : L^2(B) \to L^2(Q)$, given by

$$r(v) = v|_Q \ \text{ for } v \in H^1(B)$$

we have that the inclusion $i : H^1(Q) \to L^2(Q)$ is be given by

$$i = r \circ \iota \circ E,$$

which is compact, as it is the composition of continuous r, E with the compact embedding ι.

(c) ($\implies$) WLOG for this exercise we take Q to be $Q =]-\pi, \pi[^n$. For $u \in H^1(B)$ we have that $u \in L^2(B)$ and $\nabla u \in L^2(B)$. Thus we can expand

$$u(x) = \sum_{k \in \mathbb{Z}^n} u_k e^{ikx} \ \text{ and } \ \nabla u = \sum_{k \in \mathbb{Z}^n} \vec{d_k} e^{ikx}, \tag{18}$$

where $\vec{d_k} \in \mathbb{C}^n$ and moreover that

$$\sum_{k \in \mathbb{Z}^n} |u_k|^2 < \infty \ \text{ and } \ \sum_{k \in \mathbb{Z}^n} ||\vec{d_k}||^2 < \infty$$

We now aim to show that $\vec{d_k} = ikc_k$. Let us extend u and ∇u by periodicity on to ∂Q so we get two functions in $L^2(\bar{Q})$ with $\bar{Q} = [-\pi, \pi]^n$. Recall that $u \in H^1(\bar{Q})$ also implies that

$$\int_{\bar{Q}} \nabla u \phi \, dx = -\int_{\bar{Q}} u \nabla \phi \, dx \tag{19}$$

for all $\phi \in C_c^\infty(\bar{Q})$. Therefore let us take $\phi = e^{-ikx} \in C_c^\infty(\bar{Q})$. Then we find using integration by parts

$$\begin{aligned}
\vec{d_k} &= \int_{\bar{Q}} \nabla u e^{-ikx} \, dx \\
&= -\int_Q u \nabla(e^{-ikx}) \, dx + \underbrace{\int_{\partial Q} u e^{-ikx} \, dx}_{=0} \\
&= \int_Q iku(e^{-ikx}) \, dx \\
&= \int_Q ik \sum_{k'} u_k (e^{i(k-k')x}) \, dx \\
&= iku_k,
\end{aligned}$$

where we used swapping of sum and integral due to absolute L^2 convergence in (18). Thus from the assumption that $\sum_k \vec{d}_k$ is square summable we get that $|k|^2|u_k|^2$ is square summable, whence

$$\sum_k (1+|k|^2)|u_k|^2 < \infty.$$

($\Longleftarrow$) Now in the opposite direction we want to show that

$$\sum_k (1+|k|^2)|u_k|^2 < \infty.$$

We use for this the characterization of $W^{1,p}$ *by duality* i.e. $u \in W^{1,p}(\Omega)$ if and only if[2]

$$\left|\int_\Omega u \nabla \phi \, dx\right| \leq C||\phi||_{L^q}, \quad \text{for all } \; \phi \in C_c^\infty.$$

Now let us assume that for $u \in H^1(Q)$ we have

$$\sum_{k\in\mathbb{Z}^n} (1+|k|^2)|u_k|^2 < \infty.$$

Then again expanding $\phi = \sum_k \phi_k e^{ikx}$ and $\nabla\phi = \sum_k ik\phi_k e^{ikx}$ we get (with also expanding $u = \sum_k u_k e^{ikx}$ that

$$\begin{aligned}\left|\int_\Omega u \nabla \phi \, dx\right| &\leq \sum_{k\in\mathbb{Z}^n} |k u_k \phi_{-k}| \\ &\leq \left(\sum_{k\in\mathbb{Z}^n} |k|^2|u_k|^2\right)^{1/2} + \left(\sum_{k\in\mathbb{Z}^n} |\phi_{-k}|^2\right)^{1/2} \\ &= ||kc_k||_{\ell^2}||\phi_k||_{\ell^2}.\end{aligned}$$

But by our assumption, $||kc_k||_{\ell^2} < \infty$ from which we conclude with the duality characterization.

(d) This final statement again follows from what we have seen many times before, namely that $h^1(\mathbb{Z}^n) \hookrightarrow \ell^2(\mathbb{Z}^n)$ embeds compactly (see also e.g. exercise sheet 3 and corresponding lectures in FAII and FAI). Most definitely the expansion $\sum_{k\in\mathbb{Z}^n} u_k e^{ikx}$ should most definitely be interpreted as a Fourier sum. One can also define norms on $L^2(Q)$ and $H^1(Q)$ by pulling via $\mathcal{F}$ to $\ell^2(\mathbb{Z}^n)$ and $h^1(\mathbb{Z}^n)$. We see that the map $\iota : H^1(Q) \to L^2(Q)$ is compact. Finally as the inclusion $i : H^1_0(Q) \to H^1(Q)$ from part **(a)** is continuous we conclude that $\iota \circ i : H^1_0(Q) \hookrightarrow L^2(Q)$ is compact as the composition of a continuous and a compact function.

[2] A cute exercise to prove yourself.

We note that the final statement at the start of the exercise, i.e. the existence of a compact inclusion

$$H_0^1(\Omega) \hookrightarrow L^2(\omega)$$

is also proven. Letting the inclusion from part **(a)** be denoted by

$$io\tilde{t}a : H_0^1(\Omega) \hookrightarrow H_0^1(Q)$$

and defining the restriction $\tilde{r} : L^2(Q) \to L^2(\Omega)$ in the usual way, we conclude that

$$\tilde{r} \circ \iota \circ i \circ \tilde{\iota} : H_0^1(\Omega) \to H_0^1(Q) \to H^1(Q) \hookrightarrow L^2(Q) \to L^2(\Omega)$$

is compact as a composition of continuous maps and a compact map (ι).

10.4. Min-max characterization of eigenvalues.

We have seen in exercise sheet 3 that $-\Delta$ admits a complete orthonormal basis of eigenvectors u_m of $L^2(\Omega)$ with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_m \to \infty$. This allows for the decomposition

$$u = \sum_{m=1}^{\infty} a_m u_m. \tag{20}$$

Now for any $u \in H_0^1(\Omega)$ we have that

$$||\nabla u||_{L^2}^2 = \int_\Omega \nabla u \cdot \nabla v \; dx = \int_\Omega (-\Delta u) u \; dx$$

so expanding u as in (20) and $-\Delta u$ as

$$-\Delta u = \sum_{m=1}^{\infty} b_m u_m$$

we first want to show that

$$b_m = \lambda_m a_m. \tag{21}$$

This is easily proven using the fact that $\Delta^{-1} : L^2(\Omega) \to H_0^1(\Omega)$ is bounded, i.e. we find

$$\sum_{m=1}^{\infty} a_m u_m = u = -\Delta^{-1} \left(\sum_{k=0}^{\infty} b_m u_m \right).$$

As Δ^{-1} is bounded we know that the r.h.s. is absolutely convergent in $L^2(\Omega)$. We can therefore swap integral and Δ^{-1} and get

$$\sum_{m=1}^{\infty} a_m u_m = u = \left(\sum_{k=0}^{\infty} b_m \cdot (-\Delta^{-1})(u_m) \right) = \left(\sum_{k=0}^{\infty} \frac{b_m}{\lambda_m} u_m \right).$$

We conclude that (21) holds by orthogonality. From integration by parts it then also follows that

$$||\nabla u||_{L^2}^2 = \sum_{m=1}^{\infty} \langle a_m u_m, \lambda_m a_m u_m \rangle = \sum_{m=1}^{\infty} \lambda_m a_m^2 \underbrace{||u_m||_{L^2}^2}_{=1} = \sum_{m=1}^{\infty} \lambda_m a_m^2.$$

Hence we have that

$$\frac{||\nabla u||_{L^2}}{||u||_{L^2}} = \frac{\sum_{m=1}^{\infty} \lambda_m a_m^2}{\sum_{m=1}^{\infty} a_m^2}.$$

Now let $V \subset H_0^1(\Omega)$ be spanned by the linearly independent set $\{v_1, ..., v_k\} \in H_0^1(\Omega)$. Using Gauss elimination we can then find a $v \in V$ such that

$$v = \sum_{m=1}^{\infty} a_m u_m$$

where $a_1 = a_2 = ... = a_{k-1} = 0$ so

$$v = \sum_{m \geq k} a_m u_m.$$

Therefore we get

$$\frac{||\nabla v||_{L^2}}{||v||_{L^2}} = \frac{\sum_{m \geq k} \lambda_m a_m^2}{\sum_{m \geq k} a_m^2} \geq \frac{\lambda_k \sum_{m \geq k} a_m^2}{\sum_{m \geq k} a_m^2} = \lambda_k$$

Therefore we see that

$$\lambda_k \leq \sup_{u \in V \setminus \{0\}} \frac{||\nabla u||_{L^2}}{||u||_{L^2}}$$

On the other hand we see that if we choose $v_1 = u_1, v_2 = u_2$ and thus set $V' = \operatorname{span}\{u_1, ..., u_{m-1}\}$ we have that

$$\lambda_k = \sup_{u \in V' \setminus \{0\}} \frac{||\nabla u||_{L^2}}{||u||_{L^2}}.$$

From this we conclude as we wanted for the infimum of n-dimensional spaces $V \subseteq H^1(\Omega)$

$$\lambda_k = \inf_{\substack{V \subset H_0^1(\Omega) \\ \dim(V) = k}} \sup_{0 \neq u \in V} \frac{\|\nabla u\|_{L^2(\Omega)}^2}{\|u\|_{L^2(\Omega)}^2}.$$