11.1. A product of functions in $H_0^1(\Omega)$

For this exercise we recall that for an interval $u \in H_0^1((0,L))$ we in fact have that these u are uniformily continuous, see also exercise 2.2 for a cute proof of this fact. In particular we then have that $u \in H_0^1(\Omega)$ are characterized as continuous functions in $H^1((0,L))$ with $u(0) = 0 = u(L)$. Let us from now on only work on two variables x, y on the cube $Q = (0, L_x)_x \times (0, L_y)_y$, the higher dimensional cases are generalized to trivially.
We set $Q =$

$$f(x) = \sin\left(\frac{\pi k_x x}{L_x}\right) \quad \text{and} \quad g(y) = \sin\left(\frac{\pi k_y y}{L_y}\right). \tag{1}$$

and want to show $f \cdot g \in H_0^1(Q)$. Note that clearly we have $f \in H^1((0, L_x))$, and $g \in H^1((0, L_y))$. Now let us choose $\phi_n \in C_c^\infty(0, L_x)$ and $\psi_n \in C_c^\infty(0, L_y)$ such that

$$\lim_{n\to\infty} ||f - \phi_n||_{H^1} \to 0 \;\text{ and }\; \lim_{n\to\infty} ||g - \psi_n||_{H^1} = 0. \tag{2}$$

Clearly then $\phi_n \cdot \psi_n \in C_c^\infty((0, L_x) \times (0, L_y))$ for all $n \in \mathbb{N}$. Now we want to prove that

$$\lim_{n\to\infty} ||\phi_n \psi_n - fg||_{H^1} = 0$$

To prove this we need some triangle inequalities:

$$||\phi_n\psi_n - fg||_{H^1(Q)} \le ||\phi_n\psi_n - f\psi_n||_{H^1(Q)} + ||f\psi_n - fg||_{H^1(Q)}$$

for the L^2 part of this norm we note simply that

$$||\phi_n\psi_n - f\psi_n||_{L^2(Q)} + ||f\psi_n - fg||_{L^2(Q)} \le L_y \underbrace{||\psi_n||_{L^\infty(0,L_y)}}_{\le C_1} \underbrace{||\phi_n - f||_{L^2(0,L_x)}}_{\to 0} \tag{3}$$

$$+ L_x \underbrace{||f||_{L^\infty(0,L_X)}}_{\le C_2} \underbrace{||g - \psi_n||_{L^2(0,L_y)}}_{\to 0} \tag{4}$$

where we use that as ψ_n is convergent thus bounded in the $H_0^1(0, L_x)$ it must be bounded in the uniform norm as well. Note that we took care to seperate the integrals in the different variables on the left hand side to the right hand side. For the derivatives we can actually play the same game

$$\begin{aligned} ||\nabla(\phi_n\psi_n) - \nabla(fg)||_{L^2(Q)} &= ||\phi_n\nabla\psi_n + \psi_n\nabla\phi_n - g\nabla f - f\nabla g||_{L^2(Q)} \\ &\le \underbrace{||\phi_n\nabla\psi_n - f\nabla g||_{L^2(Q)}}_{:=I_n} + \underbrace{||\psi_n\nabla\phi_n - g\nabla f||_{L^2(Q)}}_{:=J_n} \end{aligned}$$

We then see that

$$\begin{aligned}
||\phi_n \nabla \psi_n - f\nabla g||_{L^2(Q)} &\leq ||\phi_n \nabla \psi_n - \phi_n \nabla g + \phi_n \nabla g - f\nabla g||_{L^2(Q)} \\
&\leq ||\phi_n \nabla \psi_n - \phi_n \nabla g||_{L^2(Q)} + ||\phi_n \nabla g - f\nabla g||_{L^2(Q)} \\
&\leq ||\phi_n||_{L^2(Q)} ||\nabla \psi_n - \nabla g||_{L^2(Q)} + ||\nabla g||_{L^2(Q)} ||\phi_n - f||_{L^2(Q)} \\
&= (L_y)^{1/2} \underbrace{||\phi_n||_{L^2(0,L_x)}}_{\text{bounded}} L_x^{1/2} \underbrace{||\nabla \psi_n - \nabla g||_{L^2(0,L_y)}}_{\to 0} \\
&+ L_x^{1/2} \underbrace{||\nabla g||_{L^2(0,L_y)}}_{\text{bounded}} L_y^{1/2} \underbrace{||\phi_n - f||_{L^2(0,L_x)}}_{\to 0}
\end{aligned}$$

where we use that ∇g is bounded because $g \in H_0^1(0, L_y)$ and ϕ_n bounded in L^2 norm as it converges in $H_0^1(0, L_x)$. We conclude that $I_n \to 0$ as $n \to \infty$. One can argue analogously that $J_n \to 0$ as $n \to \infty$.

11.2. Decay rate of eigenfunction expansion of $-\Delta$ on $H_0^1(\Omega)$.

First of all let us note that from corollary $C.30$ one can deduce the Riesz-Fischer theorem: for an open C^∞ domain Ω: for $u \in L^2(\Omega)$ expanded in the basis of $-\Delta : H_0^1(\Omega) \bigcap H^2(\Omega) \to L^2(\Omega)$ as

$$u(x) = \sum_{k=1}^{\infty} c_k u_k \tag{5}$$

we have that $c_k \in \ell_2(\mathbb{N})$ is given by

$$c_k = \langle u, u_k \rangle_{L^2(\Omega)}.$$

Conversely, if $c_k \in \ell^2(\mathbb{N})$ the limit $u_N = \sum_{k=1}^{N} c_k u_k$ converges to a $u \in L^2(\Omega)$.

(a) Let us first prove for $q = 1$. We then have for $u \in H^2(\Omega) \bigcap H_0^1(\Omega)$ that $-\Delta u \in L^2(\Omega)$. Let us expand

$$-\Delta u = \sum_{k=1}^{\infty} d_k u_k,$$

with $\sum_{k=1}^{\infty} |d_k|^2 < \infty$. And let u be given as in (5). Our goal is to show

$$d_k = \lambda_k c_k.$$

Note that

$$\begin{aligned}
d_k &= \langle u_k, -\Delta u \rangle_{L^2} = \langle -\Delta u_k, u \rangle_{L^2} \\
&= \lambda_k \langle u_k, u \rangle_{L^2} = \lambda_k c_k.
\end{aligned}$$

Hence we have that

$$-\Delta u = \sum_{k=1}^{\infty} \lambda_k c_k u_k \tag{6}$$

where the convergence is in $L^2(\Omega)$. In other words, setting

$$v_N = \sum_{k=1}^{N} \lambda_k c_k u_k \tag{7}$$

we have that $v_N \to -\Delta u$ in $L^2(\Omega)$ from which we have

$$\begin{aligned}
\infty > ||-\Delta u||^2_{L^2(\Omega)} &= \lim_{N\to\infty} ||v_N||^2_{L^2(\Omega)} \\
&= \lim_{N\to\infty} \langle v_N, v_N \rangle_{L^2(\Omega)} \\
&= \lim_{N\to\infty} \left\langle \sum_{k=1}^{N} \lambda_k c_k u_k, \sum_{k=1}^{N} \lambda_k c_k u_k \right\rangle_{L^2(\Omega)} \\
&= \lim_{N\to\infty} \sum_{k=1}^{N} \lambda_k^2 |c_k|^2 \\
&= \sum_{k=1}^{\infty} \lambda_k^2 |c_k|^2
\end{aligned}$$

as desired. Now for the case $q > 1$ one can play exactly the same game, by noting that if $u \in H^{2q} \cap H_0^1(\Omega)$ we have that $(-\Delta)^q u \in L^2(\Omega)$ and hence that we can expand again

$$(-\Delta)^q u = \sum_{k=1}^{\infty} \tilde{d}_k u_k,$$

and noting that

$$\begin{aligned}
\tilde{d}_k &= \langle (-\Delta)^q u, u_k \rangle_{L^2} \\
&= \langle u, (-\Delta)^q u_k \rangle_{L^2} \\
&= \lambda_k^q \langle u, u_k \rangle_{L^2} \\
&= \lambda_k^q c_k
\end{aligned}$$

And arguing as above. Conversely, let $(c_k)_{k\in\mathbb{N}}$ be such that

$$\sum_{k=1}^{\infty} |\lambda_k|^2 |c_k|^2 < \infty,$$

then $\lambda_k c_k \in \ell^2(\mathbb{N})$. Then notice that as

$$\lim_{k\to\infty} |\lambda_k| = \infty,$$

we have that there exists an M such that

$$\sum_{k=M}^{\infty} |c_k|^2 < \sum_{k=M}^{\infty} |c_k|^2 |\lambda_k|^2 \tag{8}$$

which then proves that $c_k \in \ell^2(\mathbb{N})$. This means we can then define

$$u(x) = \sum_{k=1}^{\infty} c_k u_k(x) \in L^2(\Omega) \text{ and } v(x) := \sum_{k=1}^{\infty} \lambda_k c_k u_k(x) \in L^2(\Omega),$$

where the convergence is in $L^2(\Omega)$. We now want to prove that $u \in H^2(\Omega) \cap H_0^1(\Omega)$. Let us define for $N \in \mathbb{N}$

$$u_N(x) = \sum_{k=1}^{N} c_k u_k(x) \text{ and } v_N(x) := \sum_{k=1}^{N} \lambda_k c_k u_k(x),$$

we then have that $u_N, v_N \in C^\infty(\bar{\Omega}) \cap H_0^1(\Omega)$ with the above mentioned $L^2(\Omega)$ convergence to $u, -\Delta u$ respectively. Notice that also that for all $N \in \mathbb{N}$ we have

$$-\Delta u_N = v_N.$$

Now notice that by interior regularity we have

$$||u_N||_{H^2(\Omega)} \leq C||\Delta u_N||_{L^2(\Omega)} = C||v_N||_{L^2(\Omega)}.$$

Now as v_N is a Cauchy sequence we then have that u_N is a Cauchy sequence in $H^2(\Omega)$, so $u_N \to \tilde{u}$ in $H^2(\Omega)$. As we already know that $u_N \to u \in L^2(\Omega)$ we must then also have that $u = \tilde{u}$. This argument can then easily be bootstrapped with induction for higher orders of q. In this case one defines u_N as above and v_N as

$$v_N = \sum_{k=1}^{N} \lambda_k^q c_k u_k.$$

where v_N converges again in L^2-norm to $(-\Delta)^q u$ which in $L^2(\Omega)$ has the expansion $(-\Delta)^q u = \sum_{k=1}^{N} \lambda_k^q c_k u_k$. Interior regularity then gives us

$$||u_N||_{H^{2q}} \leq C||(-\Delta)^q u_N||_{L^2(\Omega)}, \tag{9}$$

from this it then follows that u_N is Cauchy in $H^{2q}(\Omega)$ and converges to u. We conclude $u \in H_0^1(\Omega) \cap H^{2q}(\Omega)$.

(b) From the previous exercise we deduce that for $u \in H^{2q}(\Omega) \cap H_0^1(\Omega)$ with $L^2(\Omega)$ expansion as in (5) we have that

$$||\Delta^q u||_{L^2(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^{2q} |c_k|^2$$

The l.h.s of the inequality that we have to prove then follows again from interior regularity, we have that

$$||u||_{H^{2q}(\Omega)} \leq C'||\Delta^q u||_{L^2(\Omega)} = \sum_{k=1}^{\infty} |\lambda_k|^2 |c_k|^2.$$

On the other hand we have that a priori (by the sheer definition of the norm $||\cdot||_{H^{2q}}$) that

$$\sum_{k=1}^{\infty} |\lambda_k|^2 |c_k|^2 = ||\Delta^q u||_{L^2(\Omega)} \leq ||u||_{H^{2q}(\Omega)}.$$

Thus when we set $C = \max\{1, C'\}$[1] the inequality on both sides follows.

(c) By the previous exercise, we know that for $u \in H^{2q}(\Omega) \cap H_0^1(\Omega)$ expressed as (5) in the eigenvector basis of $-\Delta$ in $L^2(\Omega)$ have that we can equivalently express the norm of u in $H_0^1 \cap H^{2q}(\Omega)$ as follows

$$||u||_{H^{2q}} = \sum_{k=1}^{\infty} |c_k|^2 |\lambda_k|^{2q},$$

as the previous exercise simply implies that this sum in the series expansion defines an equivalent norm on $H^{2q}(\Omega) \cap H_0^1(\Omega)$. Now trivially, the expansion of an eigenvector u_k in this basis is simply given by $c_k = 1$ and $c_m = 0$ for all $m \neq k$. From this it follows that

$$||u_k||_{H^{2q}} = |\lambda_k|^2. \tag{10}$$

Now for $2q > n/2$ we want to apply a Sobolev embedding for case ii) in $T.30$. In the most general case (including when $k - \frac{n}{p} \in \mathbb{N}_0$) (see also e.g. theorem 6 chapter 5.6 in Evans) we see that

$$H^{2q}(\Omega) \hookrightarrow C^{2q - \left[\frac{n}{2}\right] - 1, \alpha}(\bar{\Omega}) \tag{11}$$

with $\alpha = \left[\frac{n}{2}\right] + 1 - \frac{n}{p}$ if $\frac{n}{p}$ if $\frac{n}{2}$ is not an integer, and $0 < \alpha < 1$ if $n/2$ is an integer[2]. In either case we can always trivially bind the supremum norm $||\cdot||_{L^\infty(\Omega)}$ by the Hölder norm $C^{l,\alpha}$ for any l and α. We then see with the above that

$$||u_k||_{L^\infty(\Omega)} \leq C'||u_k||_{C^{2q - \left[\frac{n}{2}\right] - 1, \alpha}} \leq C||u_k||_{H^{2q}} \leq C|\lambda_k|^2 \tag{12}$$

as required.

11.3. Asymptotics for the eigenvalues of $-\Delta$

[1] To be honest this last little step in redefining C is not that relevant for the exercise but it gives you the precise inequality as stated on the sheet.

[2] Verify these constants for yourself! It is easy to mix things up.

We know by Weyl's law (theorem $T.35$) that the eigenvalue counting function $N(T) = |\{i \in \mathbb{N} \mid \lambda_i \leq T\}|$ denote the eigenvalue counting function. Then

$$\lim_{T\to\infty} \frac{N(T)}{T^{n/2}} = \frac{\mathcal{L}^n(B_1(0))}{(2\pi)^n}\mathcal{L}^n(\Omega).$$

Now choose $T = |\lambda_n|$, we then get that

$$N(|\lambda_k|) = |\{m \in \mathbb{N} || \lambda_m| \leq |\lambda_k|\} \geq k.$$

Therefore we get after taking a power in Weyl's law

$$\limsup_{n\to\infty} \frac{k^{\frac{2}{d}}}{|\lambda_k|} \leq \lim_{k\to\infty} \frac{N(|\lambda_k|)^{\frac{2}{n}}}{|\lambda_k|} = 4\pi^2(\mathcal{L}^n(B(0,1))\mathcal{L}^n(\Omega))^{\frac{2}{n}},$$

which proves one side of the statement.
For the other half consider the adapted counting function

$$N'(T) = |\{m \mid |\lambda_m| < T\},$$

then $N(T-1) \leq N'(T) \leq N(T)$ for all T. This then implies that

$$\begin{aligned}
(2\pi)^{-n}\mathcal{L}^n(B(0,1))\mathcal{L}^n(\Omega) &= \lim_{T\to\infty} \frac{N(T-1)}{(T-1)^{\frac{n}{2}}} \frac{T^{\frac{n}{2}}}{(T-1)^{\frac{n}{2}}} \\
&= \lim_{T\to\infty} \frac{N(T-1)}{T^{\frac{n}{2}}} \\
&\leq \liminf_{T\to\infty} \frac{N'(T)}{T^{\frac{n}{2}}} \\
&\leq \limsup \frac{N'(T)}{T^{\frac{n}{2}}} \\
&\leq \lim_{T\to\infty} \frac{N(T)}{T^{\frac{n}{2}}} \\
&= \frac{\mathcal{L}^n(B_1(0))}{(2\pi)^n}\mathcal{L}^n(\Omega)
\end{aligned}$$

So Weyl's law must also hold for N'. Notice that

$$N'_\Omega(|\lambda_k|) = |\{m \in \mathbb{Z} \mid |\lambda_m| < |\lambda_k|\}| < k.$$

Therefore

$$\liminf_{k\to\infty} \frac{k^{\frac{2}{n}}}{|\lambda_k|} \geq \lim_{k\to\infty} \frac{N'(|\lambda_k|)^{\frac{2}{d}}}{|\lambda_k|} = (2\pi)^{-n}\mathcal{L}^n(B(0,1))\mathcal{L}^n(\Omega) \tag{13}$$

which concludes the proof.

11.4. Supremum bounds for eigenfunctions on compact sets.

(a) We prove this exercise using higher interior regularity. Let $\chi, \tilde{\chi}$ with $0 \leq \chi \leq 1$, $0 \leq \tilde{\chi} \leq 1$, $\Omega'' = \operatorname{supp} \chi$ and $\Omega' = \operatorname{supp} \tilde{\chi}$ on and $\tilde{\chi} \equiv 1$ on Ω''. The statement we want to prove is simply that of higher interior regularity. Note that as $f \in C^\infty(\Omega) \bigcap H^1_0(\Omega)$ that $f \in H^k_{loc}(\Omega)$ for all $k \geq 0$.
We remark that the assumption that $f \in H^k(\Omega)$ is actually slightly stronger than we need. From this it immediately follows that $u \in H^{k+2}(\Omega)$, by boundary regularity for Ω. Now, notice that multiplication $\chi \in C^\infty_c(\Omega')$ is a continuous operator from $H^{k+2}(\Omega') \to H^{k+2}(\Omega')$ as we have for any multi-index $|\alpha| \leq k+2$ that

$$||\partial_\alpha(\chi u)||_{L^2(\Omega')} \leq \sup_{|\gamma| \leq |\alpha|} ||\partial_\gamma \chi||_\infty \sup_{|\beta| \leq |\alpha|} ||\partial_\beta u||_{L^2(\Omega')} \leq C(\chi)||u||_{H^{k+2}(\Omega')}$$

Now as u solves $-\Delta u = f$ on Ω, so in particular on Ω', and $f \in H^k(\Omega')$ we can use higher regularity to estimate

$$||\chi u||_{H^{k+2}(\Omega')} \leq C(\chi)||u||_{H^{k+2}(\Omega')} \leq C(\chi)(||f||_{H^k(\Omega')} + ||u||_{L^2(\Omega')}).$$

Now as χu is supported inside Ω' we know that

$$||\chi u||_{H^{k+2}(\Omega')} = ||\chi u||_{H^{k+2}(\Omega)},$$

as we simply extend by 0. Furthermore as $\tilde{\chi} \equiv 1$ on $\operatorname{supp}(\chi) = \Omega'$ we have that

$$||f||_{H^k(\Omega')} \leq ||\tilde{\chi} f||_{H^k(\Omega)}, \tag{14}$$

and for the same reason, that

$$||u||_{L^2(\Omega')} \leq ||\tilde{\chi} u||_{L^2(\Omega)} \leq ||\tilde{\chi} u||_{H^k(\Omega)} \tag{15}$$

so the estimate follows.

(b) We know from the previous exercise using only (15) and equation (14) that

$$||\chi v||_{H^k(\Omega)} \leq C(k, \chi)(||\tilde{\chi} f||_{H^k(\Omega)} + ||\tilde{\chi} u||_{L^2(\Omega)}),$$

where $\tilde{\chi}$ is chosen as in **(a)**. As $v \in C^\infty(\Omega) \cap H^1_0(\Omega)$ was an eigenfunction, i.e. $-\Delta v = \lambda v$ we have that $f = \lambda v \in C^\infty(\Omega) \cap H^1_0(\Omega)$. In particular $f \in H^k_{loc}(\Omega)$. From this follows

$$||\chi v||_{H^k(\Omega)} \leq C(k, \chi)(|\lambda| ||\tilde{\chi} v||_{H^{k-2}(\Omega)} + ||\tilde{\chi} v||_{L^2(\Omega)}.$$

Iterating the above estimate $\frac{k}{2}$ times in the first term on the r.h.s. we have that

$$||\chi v||_{H^k(\Omega)} \leq C(k, \chi, \tilde{\chi})(|\lambda|^{k/2} + 1)||\tilde{\chi} v||_{L^2(\Omega)}$$

Under the assumption $|\lambda| > 1$ we can estimate $(|\lambda|^{k/2} + 1) \leq 2|\lambda|^{k/2})$ and absorb into the coefficient to get

$$||\chi v||_{H^k(\Omega)} \leq C(k, \chi, \tilde{\chi})|\lambda|^{k/2} ||\tilde{\chi} v||_{L^2(\Omega)} \leq C(k, \chi, \tilde{\chi})|\lambda|^{k/2} ||v||_{L^2(\Omega)}$$

(c) We recall that the equation mentioned in the hint is the Sobolev embedding into Hölder spaces for $k > \frac{n}{2}$. In particular we can choose $\frac{n}{2} < k < \frac{n}{2} + 1$ From this we know that (choosing a $\chi \in C_c^\infty(\Omega)$ with $\chi \equiv 1$ on K using exercise **(b)** that

$$||v||_{K,\Omega} \leq C||\chi v||_{H^k(\Omega} \leq C\lambda^{k/2}||v||_{L^2(\Omega)} \leq C\lambda^{\frac{n/2+1}{2}}||v||_{L^2(\Omega)}$$

and the claim follows.

11.5. The heat equation.

(a) Follows easily from arguments in previous analysis courses.

(b) Note that $u(\cdot, t) \in L^2(\Omega)$ as

$$||u(\cdot,t)||^2_{L^2(U)} = \sum_{n=1}^{\infty} |a_n|^2 e^{-2|\lambda_n|t} \leq e^{-|\lambda_1|t} \sum_{n=1}^{\infty} |a_n|^2$$

we note that from this follows that

$$\lim_{t\to\infty} ||u(\cdot,t)||_{L^2} = 0. \tag{16}$$

Now for the first claim, we compute directly

$$||u(\cdot,t) - u_0||^2_{L^2(U)} = \sum_{n=1}^{\infty} |a_n|^2 |e^{\lambda_n t} - 1|^2 \leq \epsilon + \sum_{n=1}^{N} |a_n|^2 |e^{\lambda_n t} - 1|^2$$

where $\epsilon > 0$ is given and $N \in \mathbb{N}$ is chosen large enough. We note that

$$\sum_{n=1}^{N} |a_n||e^{\lambda_n t} - 1|^2 \leq |e^{\lambda_N t} - 1|^2 \sum_{n=1}^{N} |a_n|^2 \leq |e^{\lambda_N t} - 1|^2 \sum_{n=1}^{\infty} |a_n|^2 \to 0 \text{ as } t \to 0. \tag{17}$$

where the λ_n are the eigenvalues of Δ. For the second part let us first prove the claim.

Lemma. *If $\Omega \subset \mathbb{R}^n$ is open and bounded and $f_1, f_2 ... \in H_0^1(\Omega) \bigcap C^\infty(\Omega)$ be the sequence of eigenfunctions of Δ which forms an orthonormal basis of $L^2(\Omega)$. Then let $g = \sum_{n=1}^{\infty} a_n f_n$, show that $g \in H_0^1(\Omega)$ if and only if*

$$\sum_{n=1}^{\infty} |a_n|^2 |\lambda_n| < \infty. \tag{18}$$

Proof. ($\Longrightarrow$) Assume that $g \in H_0^1(\Omega)$, we have that $|\lambda_n|^{-1/2} f_n$ forms an orthonormal basis of $H_0^1(\Omega)$. We may thus write

$$g = \sum_{n=1}^{\infty} b_n |\lambda_n|^{-\frac{1}{2}} f_n$$

where $||g||_{H^1} = \sum_{n=1}^{\infty} |b_n|^2$. Since $\iota : H_0^1(\Omega) \to L^2(\Omega)$ we have $g = \sum_{n=1}^{\infty} b_n |\lambda_n|^{-\frac{1}{2}} f_n$ in $L^2(\Omega)$ and therefore $a_n = b_n|\lambda_n|^{-\frac{1}{2}}$. In particular we have

$$\sum_{n=1}^{\infty} |a_n|^2 |\lambda_n| = \sum_{n=1}^{\infty} < \infty.$$

For the converse note that $\sum_{n=1}^{\infty} |a_n|^2 \lambda_n < \infty$ implies that the series

$$\sum_{n=1}^{\infty} ||a_n f_n||_{H^1}^2 < \infty,$$

and therefore $\sum_{n=1}^{\infty} a_n f_n$ converges in $H_0^1(\Omega)$. This convergence holds in particular when applying ι and thus it must be equal to g. Thus $g \in H_0^1(\Omega)$. $\square$

Now back to the proof, by the lemma it is enough to show that

$$\sum_{n=1}^{\infty} |a_n|^2 |\lambda_n| e^{2|\lambda_n| t} \tag{19}$$

is convergent. Then the series $\sum_{n=1}^{\infty} |a_n|^2 = ||u_0||_{L^2(\Omega)}$ is convergent, hence (a_n) is bounded and ther exists an $M > 0$ such that

$$\sum_{n=1}^{\infty} |a_n|^2 |\lambda_n| e^{2|\lambda_n| t} \leq M \sum_{n=1}^{\infty} |\lambda_n| e^{2|\lambda_n| t}.$$

By the asymptotic formula, there exist $C_1, C_2 > 0$ such that

$$C_1 n^{\frac{2}{d}} \leq |\lambda_n| t \leq C_2 n^{\frac{2}{d}}.$$

This gives us

$$\sum_{n=1}^{\infty} |\lambda_n| e^{2|\lambda_n| t} \leq \sum_{n=1}^{\infty} C_2 n^{\frac{2}{d}} e^{-2C_1 n^{\frac{2}{d}} t},$$

but the latter is convergent as it is exponentially decaying in d as $t > 0$.

(c) For any compact set $K \subset \Omega$ we have

$$\sum_{n=1}^{\infty} |a_n| e^{-|\lambda_n| t} ||f_n||_{K,\infty} \ll \sum_{n=1}^{\infty} n^{\kappa} e^{-2C_1 n^{\frac{2}{d}} t},$$

for some $\kappa > 0$, which is again a convergent series as in **(b)**. Since $C^0(K)$ is a Banach space, u is continuous on K. But K was arbirary so $u(\cdot, t)$ is continuous on Ω.
It remains to check that $||u(\cdot, t)||_{K,\infty} \to 0$ as $t \to \infty$. By the above it suffices to show that for any $\kappa > 0$

$$b(t) := \sum_{n=1}^{\infty} n^{\kappa} e^{-2C_1 n^{\frac{2}{d}} t} \to 0$$

as $t \to \infty$. Note that b is monotonely decreasing and so it suffices to show that for any $\epsilon > 0$ we can find $t \in \mathbb{R}$ with $b(t) < \epsilon$. Let $N \in \mathbb{N}$ be such that

$$\sum_{n=N}^{\infty} n^{\kappa} e^{-2C_1 n^{\frac{d}{2}}} < \epsilon.$$

Note that for any $l \in \mathbb{N}$ we have

$$\begin{aligned} b(\ell^{\frac{2}{d}}) &= \sum_{n=N}^{\infty} n^{\kappa} e^{-2C_1 n^{\frac{2}{d}} \ell^{\frac{2}{d}}} \\ &= \sum_{m \in \mathbb{N} \,:\, l | m} \left(\frac{m}{l}\right)^{\kappa} e^{2C_1 m^{\frac{2}{d}}} \\ &= \ell^{-\kappa} \sum_{m \in \mathbb{N} \,:\, l | m} m^{\kappa} e^{2C_1 m^{\frac{2}{d}}}. \end{aligned}$$

which goes to zero as $l \to \infty$.

(d) We recall that the solutoin of exercise 4 consisted of two steps: 1. Estimating the supremum norm on K by the H^k-norm for some k and
2. Estimating the H^k-norm by the eigenvalue and the L^2-norm.

For step 1 we note that we have an inclusion

$$H^k(\Omega) \to C^l(\Omega) \tag{20}$$

which for $k > l + \frac{d}{2}$ is fullfilled by the Sobolev embedding theorem. Therefore, one ought to show that

$$||\partial_\alpha f||_{K,\infty} = ||\partial_\alpha(\chi f)||_{K,\infty} \ll ||\chi f||_{H^k(\Omega)}$$

for all $\chi \in C_c^\infty(\Omega)$ with $\chi|_K \equiv 1$ and α with $|\alpha| \leq \ell$. Then applying the statment in exercise 5 for ℓ and k with $\ell + \frac{d}{2} < k \leq \ell + \frac{d}{2} + 1$ verbatim we obtain for $|\alpha| \leq \ell$. Then applying the statement in exercise 4 with $\ell + \frac{d}{2} < k \leq \ell + \frac{d}{2} + 1$ we see that

$$||\partial_\alpha f||_{K,\infty} \ll ||\chi f||_{H^k(\Omega)} \ll |\lambda|^{\frac{k}{2}} ||f||_{L^2(\Omega)} \ll |\lambda|^{\frac{d}{4} + \frac{\ell}{2} + \frac{1}{2}} ||f||_{L^2(\Omega)}.$$

This gives the polynomial rate in the eigenvalue and therefore the partial sums of $u(\cdot, t)$ converge in C^ℓ on K. Which implies smoothness for x. For smoothness in all paramters we note that the time derivatives of the partial sums of $u(\cdot, t)$ look like

$$\sum_{n=1}^{N} a_n \lambda_n^m e^{\lambda_n t} f_n(x). \tag{21}$$