12.1. The wave equation.

(a) Let us set $A(t), B(t) : H^2(\Omega) \cap H_0^1(\Omega) \to L^2(\Omega)$

$$A(t) = \cos(t\sqrt{-\Delta}) \text{ and } B(t) := \frac{\sin(t\sqrt{-\Delta}))}{\sqrt{-\Delta}}$$

We prove the statement at $t = 0$ for $A(t)$ the other cases are similar. Let us write

$$\phi_0 = \sum_{k=0}^{\infty} c_k u_k$$

where the c_k are given by

$$c_k = \langle \phi_0, u_k \rangle_{L^2}$$

Note that the estimates that were given at the start of the exercise can be sharpened with exercise 11.2, that is: for $u \in H^2(\Omega)$ we know in particular that $-\Delta u \in L^2(\Omega)$. To recap then, we know that if u has an $L^2(\Omega)$ eigenvalue expansion

$$u(x) = \sum_{k=0}^{\infty} c_k u_k, \quad \text{with} \quad c_k = \langle u, u_k \rangle_{L^2}$$

we know that $\sum_{k=0}^{\infty} |c_k| < \infty$. From the expansion $-\Delta u = \sum_{k=0}^{\infty} \lambda_k c_k u_k$ we then also have that

$$\sum_{k=0}^{\infty} \lambda_k^2 |c_k|^2 < \infty.$$

Now as $\lambda_k \to \infty$ we know that for $u \in H^2(\Omega)$, for $p < q \leq 2$ we have that

$$\sum_{k=0}^{\infty} \lambda_k^p |c_k|^2 < \sum_{k=0}^{\infty} \lambda_k^q |c_k|^2 \leq \sum_{k=0}^{\infty} \lambda_k^2 |c_k|^2 < \infty$$

as we can always estimate the tail end of the series above. We will use this fact later on in the exercise.
Now when people say use "functional calculus" in the current setting means that if you have a self-adjoint operator $A : D(A) \to H$ with point spectrum $\sigma(A) = \sigma_p(A)$, and a measurable function $f \in B^\infty(\sigma(A)$, and at the same time an orthonnormal eigenbasis w.r.t. the scalar product on H, that one can define $f(A)$ via

$$f(A)u = \sum_{k=0}^{\infty} f(\lambda_k) c_k u_k \tag{1}$$

If one wants to make this rigorous however, one will need to prove that

$$\langle f(A)u, u_k \rangle_H = f(\lambda_k) c_k,$$

which can be done with rigorously with Borel functional calculus. To give a sketch of this, we recall that for a $v \in H$ the spectral measure $\mu_{v,v}$ for some $X \in \mathcal{B}(\sigma(A))$

$$\mu_{v,v}(X) = \langle E(X)v, v\rangle_H$$

where $E : \mathcal{B}(\sigma(A)) \to \mathcal{P}(H)$ is a function mapping Borel sets of $\sigma(A)$ to projection operators $P : H \to H$. In particular, for an isolated point $\lambda_k \in \sigma(A)$ (so $\lambda_k \in \sigma_p(A)$) we have that $E(\{\lambda_k\}) = P_{\lambda_k}$ where $P_{\lambda_k} : H \to H$ is projection on the λ_k-eigenspace of A. Now clearly then we have for isolated eigenvalues $\lambda_k \in \sigma_p(A)$ that

$$\mu_{u_k,u_k}(\{\lambda_k\}) = \langle E(\{\lambda_k\})u_k, u_k\rangle_H = \langle P(\lambda_k)u_k, u_k\rangle_H = \langle u_k, u_k\rangle_H,$$

as $u_k \in V_{\lambda_k}$. Similarly for $\sigma(A) \ni \lambda \neq \lambda_k$ one can prove that $\mu_{u_k,u_k}(\{\lambda\}) = 0$, proving that $\mu_{u_k,u_k} = \delta_{\lambda_k}$ with δ_{λ_k} the Dirac measure at λ_k. Now letting $u = \sum_{k=0}^\infty c_k u_k$ and letting $X = \{\lambda_{k_j}\}_{j\in I\subseteq\mathbb{N}}$ then see that

$$\begin{aligned}\mu_{u,u}(X) &= \langle E(X)u, u\rangle_H \\ &= \left\langle \sum_{k=0}^\infty E(X)c_k u_k, \sum_{k=0}^\infty c_k u_k \right\rangle_H \\ &= \sum_{j\in I} |c_{k_j}|^2,\end{aligned}$$

proving that for arbitrary $u \in H$ we have that

$$\mu_{u,u} = \sum_{k=0}^\infty |c_k|^2 \delta_{\lambda_k}.$$

Now using the fact that for $u, v \in H$ $\mu_{u,v}$ is given by the polarization identity

$$\mu_{u,v} = \mu_{u+v} + i\mu_{u+iv} - (i+1)\mu_{u,u} - (1+i)\mu_{v,v}$$

one can show by a simple computation that for A self-adjoint with point spectrum only, we have

$$\langle f(A)u, u_k\rangle_H = \int_{\sigma(A)} f(\lambda)d\mu_{u,v}(\lambda) = \sum_{k=0}^\infty f(\lambda_k)c_k u_k,$$

as required. This shows that defining $f(A)u = \sum_{k=0}^\infty f(\lambda_k)c_k u_k$ truly is a functional calculus.

Now let us write $\phi_0 = \sum_{k=1}^\infty c_k u_k$. We then know (with the above considerations) that

$$\langle A(t)\phi_0, u_k\rangle_{L^2} = \langle \phi_0, A(t)u_k\rangle_{L^2} = \langle \phi_0, \cos(t\sqrt{\lambda_k})u_k\rangle_{L^2} = \cos(t\sqrt{\lambda_k})c_k.$$

So if $\phi_0 \in L^2(\Omega)$, we know $\sum_{j=1}^{\infty} |c_k|^2 < \infty$, hence

$$\sum_{k=1}^{\infty} |c_k \cos(t\sqrt{\lambda_k})|^2 \leq \sum_{k=1}^{\infty} |c_k|^2 < \infty, \tag{2}$$

and as we have seen before that this means exactly that

$$A(t)\phi_0 = \sum_{k=1}^{\infty} c_k \cos(t\sqrt{\lambda_k}) u_k \tag{3}$$

converges in $L^2(\Omega)$. Let us now assume $\phi_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, then using the expansion (3) we know that

$$|| \cos(t\sqrt{-\Delta})\phi_0 - \phi_0||_{H^2(\Omega)} \leq C_2 \sum_{k=1}^{\infty} | \cos(t\sqrt{\lambda_k}) - 1||c_k|^2 \lambda_k^{\frac{1}{2}} \leq 2C_2 \sum_{k=1}^{\infty} \lambda_k^{\frac{1}{2}} |c_k|^2 < \infty \tag{4}$$

where we used the first inequality given in the exercise plus a triangle inequality. The fact that the final sum on the r.h.s. is finite follows from the fact that $u \in H^2(\Omega) \cap H_0^1(\Omega)$ and thus that $\Delta u \in L^2(\Omega)$, i.e. $\sum_{k=1}^{\infty} \lambda_k^2 |c_k|^2 < \infty$ whence

$$\sum_{k=1}^{\infty} \lambda_k^{1/2} |c_k|^2 \leq \sum_{k=1}^{\infty} \lambda_k^2 |c_k|^2 < \infty$$

as $\lambda_k \to +\infty$ in the limit $k \to \infty$. As we have absolute convergence in (4) we can take the limit $t \to 0$ inside the sum and see that

$$\lim_{t \searrow 0} || \cos(t\sqrt{-\Delta})\phi_0 - \phi_0||_{H^2(\Omega)} \leq C_2 \sum_{k=1}^{\infty} \lim_{t \searrow 0} | \cos(t\sqrt{\lambda_k}) - 1||c_k|^2 = 0$$

This proves that $A(t)\phi_0 \in C^0(\mathbb{R}; H^2(\Omega))$. $B(t)\phi_1$ is proven in a similar manner.
Now let us look at $C^1(\mathbb{R}; H_0^1(\Omega))$. Continuity is then proven in the same way as above. For differentiability we have to guess the derivative of $A(t)$, which in this case of course is

$$A'(t) = -\sqrt{-\Delta} \sin(t\sqrt{-\Delta}),$$

then clearly for $\phi_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ we have $A'(t)\phi_0 \in H_0^1(\Omega)$. Moreover we now work with an $L^2(\Omega)$ expansion

$$A'(t)\phi_0 = \sum_{k=1}^{\infty} -\sqrt{\lambda_k} \sin(t\sqrt{\lambda_k}) c_k u_k, \tag{5}$$

which can again easily proven to converge in $L^2(\Omega)$ using dominated convergence and the fact that $\phi_0 \in H^2(\Omega)$. Now we prove that

$$\lim_{t \to 0} \left\| \frac{A(t) - 1}{t} \phi_0 - A'(0)\phi_0 \right\|_{H_0^1(\Omega)} = 0.$$

As $A'(0) = 0$, this reduces to proving

$$\lim_{t\to 0} \left\| \frac{A(t)-1}{t}\phi_0 \right\|_{H_0^1(\Omega)} = 0.$$

Using the first inequality again we see that

$$\left\| \frac{A(t)-1}{t}\phi_0 \right\|_{H^1} \leq C_1 \sum_{k=1}^{\infty} \left| \frac{\cos(t\sqrt{\lambda_k})-1}{t} \right|^2 |c_k|^2 \lambda_k^{1/2}$$

Using the mean value theorem we see that for $t > 0$ small there exists a $0 < \xi < t$ such that

$$\cos(t\sqrt{\lambda_k}) - 1 = -\sqrt{\lambda_k}\xi \sin(\xi\sqrt{\lambda_k}),$$

but then

$$\left| \frac{\cos(t\sqrt{\lambda_k})-1}{t} \right|^2 \leq \left| \frac{\sqrt{\lambda_k}\xi \sin(\xi\sqrt{\lambda_k})}{\xi} \right|^2 \leq \lambda_k$$

Thus

$$\left\| \frac{A(t)-1}{t}\phi_0 \right\|_{H^1} \leq C_1 \sum_{k=1}^{\infty} \left| \frac{\cos(t\sqrt{\lambda_k})-1}{t} \right|^2 |c_k|^2 \lambda_k^{1/2} \leq C_1 \sum_{k=1}^{\infty} \lambda_k^{3/2} |c_k|^2 < \infty$$

as $\Delta u \in L^2(\Omega)$. Taking the limit $t \searrow 0$ then proves the assertion. That $A'(t)$ is continuous with respect to the H^1 norm follows instantly from the expansion (5) which is absolutely convergent due to the first given estimate on the exercise sheet. The same calculations can be repeated for

$$B'(t) = \cos(t\sqrt{-\Delta}).$$

Finally we prove C^2 differentiability in L^2 in the same way, our guestimate will be that

$$A''(t) = -\Delta \cos(t\sqrt{-\Delta}).$$

We know from above that $A'(t)\phi_0$ continuous in the H^1 norm hence also in the $L^2(\Omega)$ norm. Therefore we wish to show that

$$\lim_{t\to 0} \left\| \left(\frac{A'(t)}{t} - A''(t) \right) \phi_0 \right\|_{L^2(\Omega)} = 0 \tag{6}$$

as $A'(0) = 0$. So, we see that the L^2 expansion

$$A''(t)\phi_0 = \sum_{k=1}^{\infty} -\lambda_k c_k \cos(t\sqrt{\lambda_k}) u_k$$

converges, as

$$\sum_{k=1}^{\infty} \lambda_k^2 |c_k|^2 \cos(t\sqrt{\lambda_k})^2 \leq \sum_{k=1}^{\infty} \lambda_k^2 |c_k|^2 < \infty$$

because $-\Delta\phi_0 \in L^2(\Omega)$. This immediately also proves continuity of $A''(t)\phi_0$ w.r.t the $L^2(\Omega)$-norm. Cycling back to (6), we see then with functional calculus that

$$\left\| \left(\frac{A'(t)}{t} - A''(t) \right) \phi_0 \right\|_{L^2(\Omega)}^2 = \sum_{k=1}^{\infty} \left| \frac{-\sqrt{\lambda_k}\sin(t\sqrt{\lambda_k})}{t} + \lambda_k \cos(t\sqrt{\lambda_k}) \right|^2 |c_k|^2 \tag{7}$$

Using again the mean value theorem and dominated convergence of $\sum_{k=1}^{\infty} \lambda_k^2 |c_k|^2$ as $-\Delta u \in L^2(\Omega)$ 2one can then swap again the limit in t and the sum to prove that (7) converges.

(b) This now follows from directly formally computing the derivatives of $A(t)\phi_0$ and $B(t)\phi_1$ in t (which we have shown is allowed in part **(a)**, and then deducing that

$$\begin{cases} -\partial_t^2 \phi(t,x) = -\Delta\phi(t.x) & \text{in } \; L^2(\Omega) \\ \phi(0,x) = \phi_0(x) & \text{in } \; H^2(\Omega) \cap H_0^1(\Omega) \\ \partial_t \phi(0,x) = \phi_1(x) & \text{in } \; H_0^1(\Omega) \end{cases}$$

where the equations above hold in the vector valued sense, that is with respect to the respective norms $\|\cdot\|_{L^2}, \|\cdot\|_{H^2} \|\cdot\|_{H_0^1}$

12.2. Self-adjointness of $-\Delta$ with Dirichlet boundary conditions on general domains.

Notice that $-\Delta$ on D is not well defined yet. However we define it in the only way possible namely the function sending $u \in D$ to its corresponding $f \in L^2(Omega)$. Notice that by we precisely by regularity have that D is the set $H_0^1(\Omega) \cap H^2(\Omega)$.

(a) Assuming the hint, invertibility of B follows as we know that $B = L + iK$ with $L : H_0^1(\Omega) \to H^{-1}(\Omega)$ and $K : H_0^1(\Omega) \to H^{-1}(\Omega)$ given by

$$Lu = \langle \nabla u, \nabla(\cdot) \rangle_{L^2} \quad \text{and} \quad Ku = \langle u, \cdot \rangle_{L^2}$$

invertible. Thus L is Fredholm of rank 0 and hence $L + iK$ is a compact perturbation of a rank 0 Fredholm operator hence of rank 0. Apart from this $Bu \equiv 0$ if and only if $\langle \nabla u, \nabla v \rangle = 0$ for all $v \in H_0^1(\Omega)$ and $\langle u, v \rangle_{L^2(\Omega)} = 0$ for all $v \in H_0^1(\Omega)$. This is only possible if $u = 0$ in $H_0^1(\Omega)$. Thus B is injective, and Fredholm of rank 0, hence invertible. To now prove the hint, let u_n be a bounded sequence in $H_0^1(\Omega)$. Then by Banach-Alaoglu

($H_0^1(\Omega)$ is Hilbert, thus reflexive) we have that there exists a subsequence such that $u_{n_k} \rightharpoonup u$ weakly in $H_0^1(\Omega)$ as $k \to \infty$. This means that $\langle u_{n_k}, v\rangle_{H_0^1(\Omega)} \to \langle u, v\rangle_{H_0^1(\Omega)}$ for any $v \in H_0^1(\Omega)$ and in particular $\langle u_{n_k}, v\rangle_{L^2} \to \langle u, v\rangle_{L^2}$ for any $v \in H_0^1(\Omega)$. Moreover by Rellich, we know by passing down to a furhter subsequence that $u_{n_k} \to v$ in $L^2(\Omega)$ hence $u = v$. Now we see that

$$\begin{aligned}||K(u_n - u)||_{H^{-1}(\Omega)} &= \sup_{||v||_{H_0^1(\Omega)}=1} |\langle u_{n_k} - u, v\rangle_{L^2}| \\ &\leq ||u_{n_k} - u||_{L^2(\Omega)} \underbrace{||v||_{L^2(\Omega)}}_{\leq 1} \to 0\end{aligned}$$

as $k \to \infty$. The claim follows.

(b) We first remind ourselves that $L^2(\Omega)$ embeds naturally into $H_0^1(\Omega)^* = H^{-1}(\Omega)$ via the mapping

$$\iota : L^2(\Omega) \ni \tilde{f} \mapsto \langle \tilde{f}, \cdot\rangle_{L^2} \in H_0^1(\Omega)^*.$$

As we know that $B : H_0^1(\Omega) \to H^{-1}(\Omega)$ is surjective we know that there exists a $u \in H_0^1(\Omega)$ such that

$$Bu = \langle \nabla u, \nabla(\cdot)\rangle_{L^2(\Omega)} + i\langle u, \cdot\rangle_{L^2} = \langle \tilde{f}, \cdot\rangle_{L^2}.$$

This means that $B^{-1}(f) = u \in H_0^1(\Omega)$. But then setting $f = \tilde{f} - iu \in L^2(\Omega)$ we know that we have for this u and any $v \in H_0^1(\Omega)$

$$\langle \nabla u, \nabla v\rangle_{L^2} = \langle f, v\rangle_{L^2}.$$

therefore, $u \in D$ as required. Now the fact that $-\Delta + i : D \to L^2(\Omega)$ is invertible comes from a simple computation. Let $u \in D$, and define $f \in L^2(\Omega)$

$$(-\Delta + i)u = f \tag{8}$$

Then $\iota(f) \in H^{-1}(\Omega)$ and

$$\iota(f) = \langle f, \cdot\rangle_{L^2}$$

So for all $v \in H_0^1(\Omega)$ we have

$$\begin{aligned}\iota(f)(v) &= \langle f, v\rangle_{L^2} \\ &= \langle (-\Delta + i)u, v\rangle_{L^2} \\ &= \langle \nabla u, \nabla v\rangle_{L^2} + i\langle u, v\rangle_{L^2} \\ &= Bu(v)\end{aligned}$$

We conclude that

$$(B^{-1} \circ i) \circ (\Delta + i) = Id_D$$

(c) Self-adjointness of $-\Delta + i$ follows readily, as we can repeat the above two exercises for $-\Delta - i$, i.e. by defining $B : H_0^1(\Omega) \to H^{-1}(\Omega)$ with

$$Bu(v) = \langle \nabla u, \nabla v \rangle_{L^2} - i \langle u, v \rangle_{L^2}$$

we then see that we have found the well-known sufficient criterion for self-adjointness (see the relevant chapter in the notes): namely that $-\Delta + i$ and $-\Delta - i$ are both invertible.

(d) We define $B : L^2(\Omega) \to D$ to be the left inverse of $-\Delta$, which we know exists as we know that for each $L^2(\Omega)$ there exists a unique $u \in H_0^1(\Omega)$ such that $-\Delta u = f$ weakly (i.e. $u \in D$).
So let $B \circ (-\Delta) = Id_{H_0^1}(\Omega)$. As $-\Delta$ is self-adjoint we know $\sigma_r(-\Delta) = \emptyset$. Note also that $0 \notin \sigma(-\Delta)$ We now claim that $\lambda \in \sigma(-\Delta)$ if and only if $\frac{1}{\lambda} \in \sigma(B)$. Let us prove the forward direction (the converse is the same). For the point spectrum it is clear as we know that

$$\ker(A - \lambda) = \ker(B - \frac{1}{\lambda})$$

for any $\lambda \in \mathbb{C} \backslash \{0\}$. Now assume that $\lambda \in \sigma_c(A)$ let $f \in L^2(\Omega)$. Then by assumption $-\Delta - \lambda$ has dense range in $L^2(\Omega)$ so for $-f/\lambda$ there exists a sequence $u_k \in D$ such that

$$-\Delta u_k - \lambda u_k \to -\lambda f$$

as $k \to \infty$. Set $v_k := Au_k$ so $Bv_k = u_k$ and $v_k/\lambda = -\Delta u_k/\lambda$ so we have

$$Bv_k - v_k/\lambda = u_k + \Delta u_k/\lambda \to f \text{ as } k \to \infty.$$

We conclude $1/\lambda \in \sigma_c(B)$ and the claim is proven. With this claim we are almost done, as we just remark that B is positive, hence injective as $-\Delta$ is positive. Furthermore by Rellich B factors as

$$B : L^2(\Omega) \to D \subset H_0^1(\Omega) \hookrightarrow L^2(\Omega)$$

and is compact as a map $L^2(\Omega) \to L^2(\Omega)$. An injective compact operator cannot have dense range that is unequal to $L^2(\Omega)$ as if $Bv_k \to f$ we can by boundedness of v_k always pass over to a subsequence $v_{k_j} \to v$ such that $Bv = f$. Thus $\sigma_c(B) = \frac{1}{\sigma_c(-\Delta)} = \emptyset$, proving that $\sigma(-\Delta) = \sigma_p(-\Delta)$

12.3. Spectrum of potentials on $\mathbb{R}^n$.

(a) Let $\lambda \in \mathbb{R}$ and $\lambda < 0$. Assume that $(-\Delta - \lambda)u = 0$ then

$$-\Delta u = \lambda u$$

in $H^2(\mathbb{R}^n)$ however this would imply that u is an eigenvalue with $\lambda < 0$, contradiction. Therefore $-\Delta - \lambda$ is injective. Now for $f \in L^2(\mathbb{R}^n)$ we know that the elliptic PDE

$$-\Delta u - \lambda u = f$$

has a solution $u \in H^2(\mathbb{R}^n)$, proving surjectivity.

(b) We know that $V : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact as it is multiplication operator with a compactly supported smooth function. We know from the previous exercise that $-\Delta - \lambda$ is invertible thus Fredhom of index 0. Therefore $-\Delta - \lambda + V$ is also Fredholm of index 0. Now we know that if $\lambda \in \sigma(-\Delta + V)\backslash\sigma_p(-\Delta + V)$ then $-\Delta + V - \lambda$ is injective. As it is also self-adjoint it has dense range, so $\lambda \notin \sigma_r(-\Delta + V)$. Finally again by elliptic theory we know that for $g \in L^2(\Omega)$ there exists a $u \in L^2(\Omega)$ that solves

$$-\Delta u + (V - \lambda)u = g.$$

Thus, $-\Delta + V - \lambda : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is surjective, and hence $\lambda \notin \sigma_c(-\Delta + V)$. The assertion follows.

(c) ($\Longleftarrow$) We show by contraposition that if A is self-adjoint and $\langle Au, u\rangle_H \geq 0$ then $\sigma(A) \subset [0, \infty)$. If $\lambda < 0$ we ahve

$$\begin{aligned} |\langle (A - \lambda I)u, u\rangle| &= |\langle Au, u\rangle_H - \lambda ||u||^2| \\ &\geq \langle Au, u\rangle + |\lambda|||u||^2 \\ &\geq |\lambda|||u||^2 \end{aligned}$$

We also have by Cauchy-Schwartz that $|\langle (A - \lambda I)u, u\rangle_H| \leq ||(A - \lambda I)u||_H ||u||_H$ So we have that

$$|(A - \lambda I)u||_H \geq \lambda ||u||_H$$

so $(A - \lambda I)$ is injective. Now using self-adjointness again we have that

$$\text{Range}((A - \lambda I))^\perp = \ker((A - \lambda I)^*) = \ker((A - \lambda I)) = \emptyset$$

So $A + \lambda I$ is surjective.
($\Longrightarrow$) If A is self adjoint and has $\sigma(A) \subset [0, \infty)$, then we can write

$$A = B^* B$$

for some $B : D(A) \to H$. Then we have that

$$\langle Au, u\rangle_H = \langle B^* Bu, u\rangle = ||Bu||_H^2 \geq 0.$$

(d) Take a $u \in H^2(\Omega)$ such that $\operatorname{supp} u \cap \operatorname{supp} V \neq 0$. We then see that

$$\langle -\Delta u + CVu, u\rangle_H = ||\nabla u||^2 + C \underbrace{\int_\Omega V|u|^2 \, dx}_{\leq 0}.$$

as the second quantity in the sum is lesser or equal than 0 there exists a C_0 such that when $C > C_0$,

$$||\nabla u||_{L^2}^2 + C \underbrace{\int_\Omega V|u|^2 \, dx}_{\leq 0} < 0$$

The assertion then follows from part **(c)**.