2.1. An equivalence for closed surjective operators

Let us first prove $a) \iff b)$. This follows straight from theorem $T.4$ in the notes. We have $\mathrm{Im}(A) = {}^{\perp}\ker(A^*)$. For $a) \implies b)$ notice that if $\mathrm{Im}(A) = Y$ then clearly for all $y^* \in {}^{\perp}\ker(A^*)$ it must hold that $y^*(y) = 0$ for all $y \in Y$. Hahn-Banach will then immediately leads us to the conclusion that ${}^{\perp}\ker(A^*) = \{0\}$, i.e. it can only contain the zero functional, and A^* is injective. Moreover $\mathrm{Im}(A^*)$ is closed as corollary of the Banach closed range theorem ($\mathrm{Im}(A)$ is closed and A is densely defined). For the converse we also immediately see that if A^* is injective then $\ker(A^*) = \{0\}$ hence ${}^{\perp}\ker(A^*) = Y = \mathrm{Im}(A)$ and A is surjective.
Now we prove $b) \implies c)$. If $\mathrm{Im}(A^*)$ is closed we know it is Banach as a subspace of X^*. Thus if $A^* : D(A^*) \subset Y^* \to \mathrm{Im}(A^*)$ is injective, we know that it is invertible as a map between $D(A^*) \to \mathrm{Im}(A^*)$ theorem $T.1$[1] Thus there exists a $C > 0$ such that for $A^* : \mathrm{Im}(A^*) \to D(A^*)$ we have

$$||(A^*)^{-1}x^*||_{Y^*} \leq C||x^*||_{Y^*} \tag{1}$$

and setting $c_0 = \frac{1}{C}$ and using that $x^* = A^*y^*$ for some y^* gives the result.
Finally to prove $c) \implies b)$, notice that A^* is clearly injective. Now let $x_k^* = A^*y_k^*$ for $k \in \mathbb{N}$ be a sequence in $\mathrm{Im}(A^*)$ with $x_k^* \to x^*$ as $k \to \infty$. Because of

$$c_0||y_k^*||_{Y^*} \leq \underbrace{||Ay_k^*||_{X^*}}_{:=x_k^*}, \tag{2}$$

we know that y_k^* is Cauchy. Thus let $y^* = \lim_{k\to\infty} y_k^*$. We know that A is densely defined hence A^* is a closed operator and $y^* \in D_{A^*}$.[2] We conclude $x^* = A^*y^*$ hence $\mathrm{Im}(A^*)$ is closed.

2.2. Self-adjoint extensions of $i\frac{\mathrm{d}}{\mathrm{d}t}$

(a) First of all we remark that A_α as an operator on $L^2(0,1)$ is densely defined as $D(A_\alpha)$ most definitely contains $C_c^\infty(0,1)$ which lie dense in $L^2(0,1)$. Second of all we note that A_α is symmetric on $D(A_\alpha) \subset L^2[0,1]$. This is proven by simple integration by parts: Let

[1] Originally, I wrote here that it was by the inverse mapping theorem, but this requires $D(A^*)$ to be complete (i.e. clsoed), which does not necessarily hold...
[2] Note that we cannot use the well known argument from FA I that $\mathrm{Im}(A^*)$ is closed because it is bounded from below, e.g. as in FA I ex 5.2 d), because A^* is not necessarily assumed to be bounded/continuous in this case!

$u, v \in D(A_\alpha)$ then

$$\begin{aligned}
\langle A_\alpha u, v\rangle_{L^2} &= \int_0^1 i\frac{\mathrm{d}u}{\mathrm{d}t}\overline{v}dt && (3)\\
&= \int_0^1 i\frac{\mathrm{d}u}{\mathrm{d}t}\overline{v}dt && (4)\\
&= \int_0^1 -iu\overline{\frac{\mathrm{d}v}{\mathrm{d}t}}dt + i(u(1)\overline{v(1)} - u(0)\overline{v(0)}) && (5)\\
&= \int_0^1 u\overline{\left(i\frac{\mathrm{d}v}{\mathrm{d}t}\right)}dt + \underbrace{\overbrace{e^{i\alpha}e^{-i\alpha}}^{=1}u(0)\overline{v}(0) - u(0)\overline{v(0)})}_{=0} && (6)\\
&= \langle u, A_\alpha v\rangle_{L^2}. && (7)
\end{aligned}$$

Thus given that A_α is symmetric and densely defined it makes sense to start considering its adjoint A_α^*. The easiest way to show self-adjointness is now to show that $D(A_\alpha) = D(A_\alpha^*)$. Let $\phi \in C_c^\infty((0,1)) \subset D(A_\alpha)$ and $v \in D(A_\alpha^*)$. Then we have that

$$\int_0^1 i\frac{\mathrm{d}\phi}{\mathrm{d}t}\overline{v}dt = \langle A_\alpha u, v\rangle = \langle v, A_\alpha^* u\rangle = \int_0^1 u\overline{A_\alpha^* v}dt. \tag{8}$$

From this it follows immediately that

$$A_\alpha^* v = i\frac{\mathrm{d}v}{\mathrm{d}t} \tag{9}$$

weakly in $H^1(0,1)$. Thus this means that for all $u \in D(A_\alpha)$ that

$$\begin{aligned}
\langle u, A_\alpha^* v\rangle &= \int_0^1 u\left(-i\frac{\mathrm{d}v}{\mathrm{d}t}\right) && (10)\\
&= \int_0^1 i\frac{\mathrm{d}u}{\mathrm{d}t}\bar{v}dt && (11)\\
&= \langle A_\alpha u, v\rangle && (12)\\
&= i((u(1)\overline{v(1)} - u(0)\overline{v(0)}) + \int_0^1 u\overline{A_\alpha^* v}dt && (13)\\
&= i((u(1)\overline{v(1)} - u(0)\overline{v(0)}) + \langle u, A_\alpha^* v\rangle. && (14)
\end{aligned}$$

Therefore again, we have that $v \in D(A^*)$ if and only if

$$((u(1)\overline{v(1)} - u(0)\overline{v(0)}) = 0. \tag{15}$$

But $u \in D(A)$ by assumption! Hence we have $u(1) = e^{i\alpha}u(0)$, and so for (15) to hold we have

$$\overline{v(1)} = e^{-i\alpha}\overline{v(0)}, \tag{16}$$

from which we conclude $v(1) = e^{i\alpha}v(0)$. So $v \in D(A)$, whence $D(A_\alpha) = D(A_\alpha^*)$.

(b) Surjectivity of $L : H^1(0,1) \to \mathbb{C}^2$ given by $L(u) = (u(0), u(1))$ follows, as clearly the C^∞ function $u(x) = ax + b$ for $a, b \in \mathbb{C}$ lies in $H^1(0,1)$. Given $(z_1, z_2) \in \mathbb{C}^2$ we simply choose $a = z_1$, $b = z_2$. Intuitively we know of course that $H_0^1(0,1) = \overline{C_c^\infty(0,1)}^{||\cdot||_{H^1}}$ has to be the space $\{u \in H^1(0,1) : u(0) = 0 = u(1)\}$. To prove it however we need use that H^1 convergence implies uniform convergence. In particular let ϕ_n be a sequence in $C_c^\infty(0,1)$ that converges to u in $H_0^1(0,1)$ (with respect to the H^1 norm). Then we have that for $t \in [0,1]$

$$\phi_n(t) - \phi_m(t) = \int_0^t \phi_n'(t') - \phi_m'(t') dt' \tag{17}$$

and therefore

$$|\phi_n(t) - \phi_m(t)| \leq \int_0^t |\phi_n'(t') - \phi_m'(t')| dt'. \tag{18}$$

Taking the supremum, we get

$$||\phi_n(t) - \phi_m(t)||_\infty = \sup_{t \in [0,1]} |\phi_n(t) - \phi_m(t)| \tag{19}$$

$$\leq \sup_{t \in [0,1]} \int_0^t |\phi_n'(t') - \phi_m'(t')| dt' \tag{20}$$

$$= \int_0^1 |\phi_n'(t') - \phi_m'(t')| dt' \tag{21}$$

$$= ||\phi_n' - \phi_m'||_{L^2} \underbrace{||1||_{L^2}}_{=1}, \tag{22}$$

where we use a Cauchy-Schwarz (Hölder) inequality in the last part. Now as ϕ_n converges in the H^1 norm we have that it is a Cauchy sequence and in particular that $||\phi_n' - \phi_m'||_{L^2} \to 0$ when $m, n \to \infty$. From this it follows that ϕ_n is a Cauchy sequence in the uniform norm $|| \cdot ||_\infty$ and thus it converges, to a continuous function which is u by the uniqueness of the limit. We find that $u(0) = 0 = u(1)$ and the claim follows. We finally find that quotient map $p : H^1(0,1)/H_0^1(0,1) \to \mathbb{C}^2$ induced by L is an isomorphism, whence $\dim(H^1(0,1)/H_0^1(0,1)) = 2$.

(c) From the hint it is immediately clear that neither B_0 or B_0^* are self-adjoint as $D(B_0) = H_0^1(0,1)$ and $D(B_0^*) = H^1(0,1)$ and $H_0^1(0,1) \neq H^1(0,1)$.[3] Additionally, know that for any self-adjoint extension B of B_0 we have $D(B_0) \subseteq D(B)$ and $D(B^*) \subseteq D(B_0)$ and $D(B) = D(B^*)$. With this and the fact that neither B_0 and B_0^* are strict inequalities we get the following chain of strict inclusions

$$H_0^1(0,1) = D(B_0) \subsetneq D(B) \subsetneq D(B_0^*) = H^1(0,1). \tag{23}$$

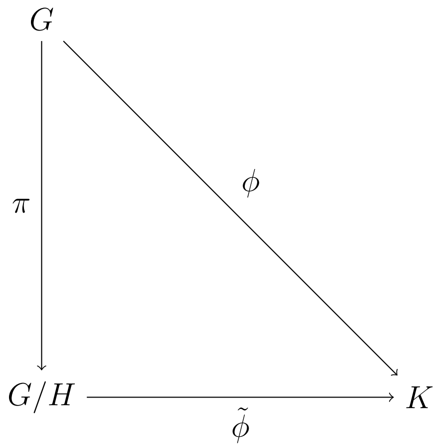
[3]Actually, this was also discussed in the lecture.

From this it follows that

$$\dim D(B_0)/D(B) < \dim D(B_0^*)/D(B) < \dim D(B_0^*)/D(B_0). \tag{24}$$

It is clear that $\dim D(B_0)/D(B) = 0$ and from part b) we know that $\dim D(B_0^*)/D(B_0) = 2$. Therefore it must follow that $\dim D(B_0^*)/D(B) = 1$ and $D(B_0^*)/D(B) \cong \mathbb{C}$. Let $q : D(B_0^*) \to D(B^*)/D(B)$ be the quotient map. Note that $D(B_0) \subset D(B) = \ker q$.

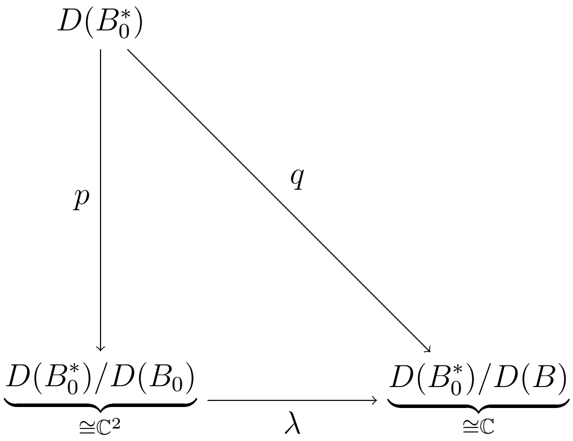
Let us now recall[4] the universal property of quotients: if we have a groups G and K and a group homomorphism $\phi : G \to K$, a normal subgroup $H \triangleleft G$ satisfying $H \subseteq \ker \phi$ with quotient map $\pi : G \to G/H$, then there exists a $\tilde{\phi} : G/H \to K$ such that the following diagram commutes:



. That is, we have $\phi = \tilde{\phi} \circ \pi$

Now in our case we take $G = D(B_0^*)$, $K = D(B_0^*)/D(B)$ and $\phi = q : D(B_0^*) \to D(B_0^*)/D(B)$. Remember that from the previous exercise we have the induced quotient map $p : H^1(0,1) = D(B_0^*) \to H^1(0,1)/H_0^1(0,1) = D(B_0^*)/D(B_0) \cong \mathbb{C}^2$ given explicitly by $p(u) = (u(0), u(1))$. Note that $D(B_0) \subset D(B)$, so $D(B_0) \subset \ker q$. Thus we take $H = D(B_0)$ and we deduce the existence of a $\lambda : D(B_0^*)/D(B_0) \to D(B_0^*)/D(B)$ such that the following diagram commutes:

[4] Yes, we are actually going to use some abstract algebra...



, i.e. we have that

$$q(u) = \lambda(p(u)) = \lambda(u(0), u(1)). \tag{25}$$

We conclude that

$$\begin{aligned} D(B) &= \ker \lambda \circ p \\ &= \{u \in H^1(0,1) \ : \ p(u) \in \ker \lambda\} \\ &= \{u \in H_0^1(0,1) \ : \ \lambda(u(0), u(1)) = 0\}, \end{aligned}$$

as desired.

(d) We conclude from parts a) to c) that if B is a self-adjoint extension of B_0 then

$$D(B) = \{u \in H^1(0,1) \ : \ \lambda(u(0), u(1)) = 0\}. \tag{26}$$

As λ is in fact a linear functional from $\mathbb{C}^2$ to $\mathbb{C}$ this implies that there exists constants $a_0, a_1 \in \mathbb{C}$ such that

$$D(B) = \{u \in H^1(0,1) \ : \ a_0 u(0) + a_1 u(1) = 0\}. \tag{27}$$

Now if $a_0 = 0$ and $a_1 = 0$ we get that $D(B) = H_0^1(0,1)$, hence B cannot be adjoint. If WLOG only $a_0 = 0$ we have that $D(B) = \{u \in H^1(0,1) \ : \ u(1) = 0\}$. Integrating by parts then we see that for $v \in D(B^*)$ similar as in example $E.8$ that for $v \in D(B)$

$$\begin{aligned} \langle u, Bv \rangle &= \int_0^1 u \overline{iv'} dt \\ &= i(\underbrace{u(1)\overline{v(1)}}_{=0} - u(0)\overline{v(0)}) = \int_0^1 iu'\bar{v} dt \end{aligned}$$

while we also have per definition

$$\langle u, Bv\rangle = \langle B^* u, v\rangle = \int_0^1 iu'\bar{v}dt. \tag{28}$$

from which we deduce that for $u(0)$ arbitrary that $\overline{v(0)} = 0$, i.e. that $v(0) = 0$. Therefore in this case we must have $D(B^*) = \{u \in H^1(0,1) \ : \ u(0) = 0\}$, $D(B) \neq D(B^*)$ and B cannot be self-adjoint. Thus assume $a_0 \neq 0$ and $a_1 \neq 0$, we then get

$$D(B) = \{u \in H^1(0,1) \ : \ \frac{a_0}{a_1} u(0) + u(1) = 0\}. \tag{29}$$

By a similar integration as above, that

$$\langle Bu, v\rangle = i(u(1)\overline{v(1)} - u(0)\overline{v(0)}) + \langle u, Bv\rangle. \tag{30}$$

For B to be self-adjoint we want the boundary terms to vanish, we get for $u, v \in D(B)$:

$$u(1)\overline{v(1)} - u(0)\overline{v(0)} = 0. \tag{31}$$

We use

$$u(1) = -\frac{a_0}{a_1} u(0) \text{ and } \overline{v(1)} = -\frac{\overline{a_0}}{\overline{a_1}}\overline{v(0)} \tag{32}$$

and thus we have that

$$u(0)\overline{v(0)}\left(\frac{a_0}{a_1}\frac{\overline{a_0}}{\overline{a_1}} - 1\right) = u(0)\overline{v(0)}\left(\left|\frac{a_0}{a_1}\right|^2 - 1\right) = 0. \tag{33}$$

This last equation can only hold if for all $u, v \in D(B)$ if $\left|\frac{a_0}{a_1}\right| = 1$, i.e. $-\frac{a_0}{a_1} = e^{i\alpha}$ for some α. We then conclude that

$$D(B) = \{u \in H^1(0,1) \ : \ u(1) = e^{i\alpha} u(0)\} = D(A_\alpha). \tag{34}$$

2.3. Spectrum and adjoint of an operator on $l^2(\mathbb{Z})$.

(a) Assume $(a^{(m)}, Aa^{(m)})_{m\in\mathbb{N}}$ converges in the graph of $\Gamma_A \subset \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$ to some (a, b) as $m \to \infty$, that is

$$\lim_{m\to\infty} ||a^{(m)} - a||_{\ell^2} = 0 \ \text{ and } \ \lim_{m\to\infty} ||Aa^{(m)} - b||_{\ell^2} = 0, \tag{35}$$

where we denote with $||\cdot||_{\ell^2}$ the standard ℓ^2-norm

$$||u||_{\ell^2} = \left(\sum_{n\in\mathbb{Z}} |u_n|^2\right)^{1/2} \tag{36}$$

From the first equality and the fact that convergence in (36) (i.e. absolute convergence) implies pointwise (entry-wise) convergence we know that for all $n \in \mathbb{N}$,

$$\lim_{m\to\infty} |a_n^{(m)} - a_n| = 0. \tag{37}$$

Of course from this it follows easily that for each $q_n \in \mathbb{C}$

$$\lim_{m\to\infty} |q_n a_n^{(m)} - q_n a_n| = \lim_{m\to\infty} |q_n||a_n^{(m)} - a_n| = 0, \tag{38}$$

from which pointwise convergence follows

$$\lim_{m\to\infty} q_n a_n^{(m)} = q_n a_n \tag{39}$$

on the other we have absolute convergence from (35)

$$\lim_{m\to\infty} \left(\sum_{n\in\mathbb{Z}} |q_n a_n^{(m)} - b_n|^2 \right)^{1/2} = 0. \tag{40}$$

As again absolute convergence in ℓ^2 implies pointwise convergence we end up with the fact

$$b_n = q_n a_n \text{ for all } n \in \mathbb{N}. \tag{41}$$

We conclude $b = Aa$ and that $b \in \ell^2(\mathbb{Z})$ by completeness, therefore $a \in D(A)$ and we are done.

Remark 1. *The above statement is a specific case of a more general principle that holds, namely that multiplication operators on measurable spaces are always closed. In particlar the following holds for $1 \leq p \leq \infty$: if (X, μ) is a measurable space and $m : X \to \mathbb{C}$ is a measurable function then the operator $M : \mathcal{L}^p(X,\mu) \to \mathcal{L}^p(X,\mu)$ given by*

$$M(f)(x) := m(x)f(x) \text{ on } D(M) := \{f \in \mathcal{L}^p(X,\mu) : m \cdot f \in \mathcal{L}^p(X,\mu)\} \tag{42}$$

is always a closed operator. It might be a fun exercise to prove this more general fact as well.

(b) First of all (!!!) we need to make sure an adjoint can be well-defined in the first place. In other words we need to make sure that D is densely defined in $\ell^2(\mathbb{Z})$ as we recall that in general A^* is *not* unique if $D(A)$ is not densely defined. One could conclude this directly from the fact that $c_{00} \subset D$ lies dense in $\ell^2(\mathbb{Z})$. One could also prove this with the following reasoning (which can also be applied to general arguments in $\mathcal{L}^p(X,\mu)$)[5]:

[5] Can you think of reason why the former argument fails in $L^2(\mathbb{R})$ for multiplication with a measurable function? I.e. using density of $C_c^\infty(\mathbb{R})$?

assume that D does not lie dense in $\ell^2(\mathbb{Z})$ then $\dim(D^\perp) \geq 1$. Assume that $g \in D^\perp \setminus \{0\}$, we then define the sequence h entry-wise as

$$h_n := \frac{1}{1+|q_n|^2}\overline{g}_n. \tag{43}$$

Clearly then, we have that $h \in \ell^2(\mathbb{Z})$ as $\frac{1}{1+|q_n|^2} \leq 1$ for all $n \in \mathbb{Z}$ and g was assumed to be in $\ell^2(\mathbb{Z})$. However we also have $h \in D$ as $|z| \leq (1+|z|^2)$ for all $z \in \mathbb{C}$, whence

$$||q \cdot h||_{\ell^2}^2 = \sum_{n\in\mathbb{Z}} \frac{|q_n|^2}{(1+|q_n|^2)^2}|g_n|^2 \leq \sum_{n\in\mathbb{Z}} |g_n|^2 = ||g||_{\ell^2} < \infty. \tag{44}$$

Now combining this with the fact that $g \in D^\perp$, we find that

$$0 = \langle g, h\rangle_{\ell^2} = \sum_{n\in\mathbb{Z}} \frac{1}{1+|q_n|^2}|g_n|^2, \tag{45}$$

from which we conclude that $g_n = 0$ for all $n \in \mathbb{N}$, leading to a contradiction.

The adjoint of A determined through the equation $(Ax, y)_{\ell^2} = (x, A^*y)_{\ell^2}$ for $x, y \in \ell^2(\mathbb{Z})$. From this we derive that

$$(Ax, y)_{\ell^2} = \sum_{n\in\mathbb{Z}} (q_n a_n)\overline{b_n} = \sum_{n\in\mathbb{Z}} a_n \overline{(\overline{q_n} b_n)} = (x, A^*y)_{\ell^2}. \tag{46}$$

From this it readily follows that the action of A^* should be given with by

$$A^* y_n = \overline{q_n} y_n \text{ for all } y \in \ell^2(\mathbb{Z})^* = \ell^2(\mathbb{Z}), \tag{47}$$

clearly with domain

$$D(A^*) = \{a \in \ell^2(\mathbb{Z}) : (\overline{q_n} a_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})\}, \tag{48}$$

because when we set $z_n = \overline{q_n} a_n$ for all $n \in \mathbb{Z}$ we have $(z_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ and clearly

$$(x, A^*y)_{\ell^2} = (x, z)_{\ell^2} \text{ for all } x \in D. \tag{49}$$

The final statement, namely that A is self-adjoint, if $(q_n)_{n\in\mathbb{Z}} \subset \mathbb{R}$ follows immediately from (46) and the fact that $\overline{q_n} = q_n$ for all $n \in \mathbb{Z}$.

(c) It is a general known fact that the spectrum $\sigma(A)$ is a closed set (see FA I, lecture 12)[6]. Clearly we know that the $e^i \in \ell^2(\mathbb{Z})$ given by

$$(e^i_n)_{n\in\mathbb{Z}} = \delta_{in} \tag{50}$$

[6] For perhaps a more concrete proof of this fact you can argue that the resolvent set is open by expanding in a Neumann series

are eigenvectors of A with eigenvalue $q_i \in \mathbb{C}$

$$Ae^i = q_i e^i. \tag{51}$$

Therefore the entries of q must be part of the point spectrum of A and we know that

$$\overline{\{q_n | n \in \mathbb{Z}\}} \subseteq \sigma(A). \tag{52}$$

We claim that in fact equality holds. Let $\lambda \in \mathbb{C} \setminus \overline{\{q_n \ : \ n \in \mathbb{Z}\}}$. Then clearly $(\lambda \mathbf{1} - A)^{-1} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is given directly by

$$A^{-1} a_n := \frac{1}{\lambda - q_n} a_n, \tag{53}$$

and it is bounded as

$$C = \inf_{n \in \mathbb{Z}} |\lambda - q_n| > 0 \tag{54}$$

implies

$$||A^{-1} a_n||_{\ell^2} = \sum_{n \in \mathbb{Z}} \left(\frac{1}{|\lambda - q_n|^2} |a_n|^2 \right)^{1/2} \leq \frac{1}{C} ||a||_{\ell^2}. \tag{55}$$

From which we conclude that $\lambda \notin \sigma(A)$.

2.4. The adjoint operator is *always* closed

Let $V : H \times H \in H \to H \times H$ be given as in the hint, i.e. $V(x, y) = (-y, x)$. Clearly V is an isometric isomorphism. We will now show that $\Gamma_{A^*} = (V(\Gamma_A))^\perp$.
For $\Gamma_{A^*} \subseteq [V(\Gamma_A)]^\perp$ we not that

$$\Gamma_{A^*} = \{(x, A^* x) \in H \times H : x \in D(A^*)\},^7 \tag{56}$$

Thus if $x \in D(A^*)$ and $y \in D(A)$, and letting $\langle \cdot, \cdot \rangle_H$ induce the scalar product on $H \times H$ we have

$$\begin{aligned} \langle (x, A^* x), V(y, Ay) \rangle_{H \times H} &= \langle (x, A^* x), (-Ay, y) \rangle_{H \times H} & (57) \\ &= \langle x, -Ay \rangle_H + \langle A^* x, y \rangle_H & (58) \\ &= -\langle x, Ay \rangle_H + \langle x, Ay \rangle_H = 0. & (59) \end{aligned}$$

To see that $[V(\Gamma_A]^\perp \subseteq \Gamma_{A^*}$, let $(x, y) \in (V(\Gamma_A))^\perp$. Then if $z \in D(A)$ we have

$$0 = \langle (x, y), (-A(z), z) \rangle_{H \times H} = -\langle x, A(z) \rangle_H + \langle y, z \rangle_H. \tag{60}$$

Therefore $\langle A(z), x \rangle_H = \langle z, y \rangle_H$. Therefore for $x \in H$ there exists a $y \in H$ such that $A^*(x) = l_y^* = \langle \cdot, y \rangle_H$. We conclude $x \in D(A^*)$ and we are done. The statement that an adjoint operator is always closed now follows from closedness of annihilator sets.

[7] Note that we identify H^* with itself using Riesz representation theorem, hence dropping the x^* notation.