NOTE: this solution sheet went through quite some iterations. Please do not hesitate to contact me at pieterbart.peters@math.ethz.ch if you have any questions, or found some errors. There might be some minor corrections somewhere at the end of the semester so feel free to check it out at a later point (you can see the date modified at the bottom of the page).

3.1. Spectrum of a self-adjoint operator on $H^2(\mathbb{S}^1)$

(a) We recall (see e.g. FA I, lecture 10 example 6) that on $\mathbb{S}^1$ the Sobolev spaces for $s \in \mathbb{R}$ are given by

$$H^s(\mathbb{S}^1) = \mathcal{F}^{-1}(h^s(\mathbb{Z})) \text{ where } h^s(\mathbb{Z}) = \left\{ a := ((a_n)_{n\in\mathbb{Z}}) \ : \ \left(\sum_{n\in\mathbb{N}} (1+|n|)^{2s} |a_n|^2 < \infty \right)^{1/2} \right\}, \tag{1}$$

In other words

$$H^s(\mathbb{S}^1) = \{u \in L^2(\mathbb{S}^1) : \sum_{n\in\mathbb{Z}} (1+|n|)^{2s} |\mathcal{F}u(n)| < \infty\}, \tag{2}$$

with norm

$$||u||_{H^s} = \left(\sum_{n\in\mathbb{Z}} (1+|n|)^{2s} |\mathcal{F}u(n)|^2 \right)^{1/2}, \tag{3}$$

[1]

and scalar product

$$\langle u, v\rangle_{H^s} = \left(\sum_{n\in\mathbb{Z}} (1+|n|)^{2s} \mathcal{F}u(n)\overline{\mathcal{F}v(n)} \right). \tag{4}$$

As such, functions in $H^s(\mathbb{S}^1)$ can always be identified by their Fourier series,

$$u \in H^s(\mathbb{S}^1) \text{ iff } u(\theta) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta} \text{ with } (a_n)_{n\in\mathbb{Z}} \in h^s(\mathbb{Z}). \tag{5}$$

In the above it obviously holds that $a_n = \mathcal{F}u(n)$, where we recall that the forward Fourier transform $\mathcal{F} : H^2(\mathbb{S}^1) \to h^2(\mathbb{Z})$ is given by

$$((\mathcal{F}u)_n)_{n\in\mathbb{Z}} = (\mathcal{F}u(n))_{n\in\mathbb{Z}} \ \text{ with } \ \mathcal{F}u(n) = \int_{\mathbb{S}^1} u(\theta) e^{-in\theta} d\theta \tag{6}$$

and the inverse transform $\mathcal{F}^{-1} : h^s(\mathbb{Z}) \to H^s(\mathbb{S}^1)$ given by $(a_n)_{n\in\mathbb{Z}}$ by

$$\mathcal{F}^{-1}((a_n)_{n\in\mathbb{Z}}) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}. \tag{7}$$

[1] Alternatively, one can take the closure of $C^\infty(\mathbb{S}^1)$ with respect to this norm.

As the norm of $H^s(\mathbb{S}^1)$ is defined by pulling back the norm from $h^s(\mathbb{Z})$ both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are isometric (unitary!) isomorphisms. We will make use of this unitary equivalence several times throughout this exercise.
First we treat the hint and show that $D_\theta^2 \pm i : H^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is invertible. This is most easily seen when we consider the unitarily equivalent operator $\hat{D}_\theta^2 : h^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, defined via $\hat{D}_\theta^2 := \mathcal{F} \circ \hat{D}_\theta^2 \circ \mathcal{F}^{-1}$. A simple computation yields for $(a_n)_{n\in\mathbb{N}} \in h^2(\mathbb{S}^1)$ that $\hat{D}_{theta}^2$ is explicitly given by

$$\mathcal{F}^{-1} \circ D_\theta^2 \circ \mathcal{F}(a_n)_{n\in\mathbb{N}} = \mathcal{F}\left(\sum_{n\in\mathbb{Z}} D_\theta^2 a_n e^{in\theta}\right) \tag{8}$$

$$= \mathcal{F}\left(\sum_{n\in\mathbb{Z}} n^2 a_n e^{in\theta}\right) \tag{9}$$

$$= (n^2 a_n)_{n\in\mathbb{Z}}. \tag{10}$$

Note that from the above clearly follows that $\hat{D}_\theta^2 : h^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is bounded and thus also $D_\theta^2 : H^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$

Then, we have that $\hat{D}_\theta^2 \pm i$ is given by

$$(\hat{D}_\theta^2 \pm i)(a_n)_{n\in\mathbb{Z}} = ((n^2 \pm i)a_n)_{n\in\mathbb{Z}}. \tag{11}$$

Clearly we can explicitly invert this when we define $(\hat{D}_\theta^2 \pm i)^{-1} : \ell^2(\mathbb{Z}) \to h^2(\mathbb{Z})$ as

$$(\hat{D}_\theta^2 \pm i)^{-1}(a_n)_{n\in\mathbb{Z}} := ((n^2 \pm i)^{-1} a_n)_{n\in\mathbb{Z}}, \tag{12}$$

which is bounded as well. We conclude that the unitarily equivalent operator $D_\theta^2 \pm i$ also has a bounded inverse. Secondly we analyze the multiplication operator $V : H^2(\mathbb{Z}) \to L^2(\mathbb{Z})$. This is most easy to do without Fourier transforming. Let us define $\tilde{V} : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ (a slightly larger domain) via

$$\tilde{V}(u) = V \cdot u. \tag{13}$$

Then we have that

$$||\tilde{V}u||_{L^2}^2 = \int_{\mathbb{S}^1} |V(x)|^2 |u(x)|^2 d\mu(x) \leq ||V||_{L^\infty}^2 ||u||_{L^2}^2 \tag{14}$$

thus $\tilde{V} : L^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ is bounded. [2] Moreover we have seen in class that the compact inclusion $\iota : H^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is compact. Using this Sobolev embedding we conclude that $V = \tilde{V} \circ \iota$ is compact as the composition of a bounded operator and compact inclusion.

[2] If one were to prove this on the Fourier side for $\hat{\tilde{V}} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ one would have to use Young's inequality for convolutions (as the Fourier transform sends products to convolutions)

Thus we see that $D_\theta^2 + V \pm i : H^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is a Fredholm operator of index 0 as it is the sum of $D_\theta^2 \pm i : H^2(\mathbb{Z}) \to L^2(\mathbb{Z})$, which is invertible and $V : H^2(\mathbb{Z}) \to L^2(\mathbb{Z})$ which is compact, see also FA I chapter 11. As a reminder, left and right multiplying by $(D_\theta^2 \pm i)^{-1}$ we have

$$\begin{aligned} ((D_\theta^2 \pm i)^{-1} \circ (D_\theta^2 \pm i + V) &= Id_{H^2} + (D_\theta^2 \pm i)^{-1} \circ V && (15)\\ &= Id_{H^2} + K_1 && (16)\end{aligned}$$

and

$$\begin{aligned} (D_\theta^2 \pm i + V) \circ (D_\theta^2 \pm i)^{-1} &= Id_{L^2} + V \circ (D_\theta^2 \pm i)^{-1} && (17)\\ &= Id_{L^2} + K_2, && (18)\end{aligned}$$

where $K_1 := (D_\theta^2 \pm i)^{-1} \circ V : H^2(\mathbb{S}^1) \to H^2(\mathbb{S}^1)$ and $K_2 := (D_\theta^2 \pm i + V) \circ (D_\theta^2 \pm i)^{-1} : L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ are again compact operators as compositions of compact and bounded operators.

Finally we know that $P = D_\theta^2 + V : H^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is symmetric which follows directly[3] from

$$\begin{aligned} \langle Pu, u\rangle_{L^2} &= \int_{\mathbb{S}^1} D_\theta^2 u \cdot \overline{u} d\theta + \int_{\mathbb{S}^1} Vu \cdot \overline{u}\, d\theta && (19)\\ &= \int_{\mathbb{S}^1} |D_\theta u|^2\, d\theta + \int_{\mathbb{S}^1} u\overline{Vu}\, d\theta && (20)\\ &= \int_{\mathbb{S}^1} u\overline{D_\theta^2 u}\, d\theta + \int_{\mathbb{S}^1} u\overline{Vu}\, d\theta = \langle u, Pu\rangle_{L^2}. && (21)\end{aligned}$$

We know therefore that $\sigma_p(P) \subset \mathbb{R}$ and therefore that $\pm i \notin \sigma_p(H)$ and thus that $P \pm i = D_\theta^2 + V + i$ is injective. As P is also Fredholm of index 0, it is surjective and therefore bijective with continuous inverse. By theorem $T.5(iii)$, H is self-adjoint.

(b) The easiest way to prove that $\sigma(P) = \sigma_p(P)$ and has a complete orthonormal basis, is done by "raising the potential" V and then use some results that we know from chapter 13, FA I.
Let us choose a $z \in \mathbb{R}_+$ with $z > ||V||_{L^h\infty} + 1$ Then it is easy to see that (see also chapter 13, FA I, page 4) $P + z : H^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$ is injective. Assume $u \in \ker(P + z)$ then

$$\begin{aligned} \langle (P+z)u, u\rangle_{L^2} &= ||D_\theta u||_{L^2}^2 + \int_{\mathbb{S}^1} V|u|^2 + z|u|^2 d\theta && (22)\\ &\geq \int_{\mathbb{S}^1} (z - |V|)|u|^2 d\theta \qquad \geq (z - ||V||_{L^\infty})||u||_{L^2}^2 && (23)\\ &> ||u||_{L^2}^2 \geq 0. && (24)\end{aligned}$$

[3]This is arguably even easier to show in the Fourier domain

One can similarily to question a) very easily prove that $P + z$ is a self-adjoint operator, hence by equation (22) $P + z$ is bounded from below. Therefore we know from an exercise sheet in FA I, that if $P + z$ is injective then $(P + z)^* = P + z$ is also surjective. Therefore $P + z$ is bijective, bounded and therefore its resolvent $R_{-z} = (z + P)^{-1}$ lies in $L(L^2(\mathbb{S}^1), H^2(\mathbb{S}^1)$ by the theorem of bounded inverse. Let us consider the same resolvent $\tilde{R}_{-z}$ as a map from $L^2(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$. Then $\tilde{R}_{-z}$ is also bounded. Therefore $R_{-z} = \tilde{R}_{-z} \circ \iota$ is compact again as a composition of a compact embedding and a bounded map. It is self-adjoint because $P+z$ is self-adjoint. Therefore, we can apply the theorems in chapter 13 of FA I, for compact self-adjoint operators. By the first theorem in the chapter we know that

$$\sigma(R_{-z})\backslash\{0\} = \sigma_p(P)\backslash\{0\}. \tag{25}$$

As R_{-z} is self-adjoint, we therefore know that $\sigma(R_{-z}) \subset \mathbb{R}$. Finally, R_{-z} is positive because $P + z$ is: if we let $v \in L^2(\mathbb{S}^1)\backslash\{0\}$, and set $u = R_{-z}v$ then

$$\begin{aligned} 0 &< \langle (z + P)u, u\rangle_{L^2} && (26)\\ &= \langle (z + P)R_{-z}v, R_{-z}v\rangle_{L^2} && (27)\\ &= \langle v, R_{-z}v\rangle_{L^2} && (28)\\ &= \langle R_{-z}v, v\rangle_{L^2}. && (29)\end{aligned}$$

Therefore the Courant-Fischer min-max principle applies to yield a set of eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_k \searrow 0$ as $k \to \infty$ with eigenvectors $\{v_k\}_{k\in\mathbb{N}}$ that span $L^2(\mathbb{S}^1)$, $\ker(R_{-z}) = \emptyset$. But then we have for this set v_k that

$$(P + z)v_k = \lambda_k^{-1} v_k \tag{30}$$

and hence

$$Pv_k = (\lambda_k^{-1} - z)v_k. \tag{31}$$

Clearly, the eigenvalues $\lambda_k^{-1} - z$ can only accumulate at $\pm\infty$ and we are done.

3.2. Spectral calculus for commuting self-adjoint operators.

By the Borel functional calculus, we know that there exists a $*$-algebra homomorphism $\phi : \mathcal{B}^\infty(\sigma(A)) \to L(H)$ such that if $B \in L(H)$ commutes with $A \in L(H)$ then B commutes with $\phi(f) := f(A)$ for all $f \in \mathcal{B}^\infty(\sigma(A))$, i.e.

$$f(A) \circ B = B \circ f(A). \tag{32}$$

Keeping in mind that ϕ maps Borel measurable functions on $\sigma(A)$ into continuous operators in $L(H)$, we can therefore apply the same trick again: i.e. for $f(A) \in L(H)$

and $B \in L(H)$ there exists a $\tilde{\phi} : \sigma(B) \to L(H)$, *-algebra homomorphism such that if $f(A)$ commutes with B then $\tilde{\phi}(g) := g(B)$ commutes with $f(A)$ for all $g \in \mathcal{B}^\infty$, so we have

$$f(A) \circ g(B) = g(B) \circ f(A) \tag{33}$$

for all $f \in \mathcal{B}^\infty(\sigma(A))$ and $g \in \mathcal{B}^\infty(\sigma(B))$ and in particular for $f, g \in \mathcal{B}^\infty(\mathbb{R})$ as well (as they are Borel measurable on both $\sigma(A)$ and $\sigma(B)$ respectively. Finally, the above statement holds in particular for spectral projects $f = \mathbf{1}_\Omega$ for $\Omega \subset \sigma(A)$ measurable and $g = \mathbf{1}_{\Omega'}$ with $\Omega' \subset \sigma(B)$ measurable.

3.3. Resolvents to characterize the spectral measure.

We know from Riesz-Markov that for $u \in H$ $d\tilde{\mu}_u$ and $A \in L(H)$ self-adjoint there exists a regular Borel measure (the spectral measure of u) μ_u (supported on $\sigma(A)$ such that

$$(f(A)u, u) = \int_{\sigma(A)} f(t) d\mu_u(t). \tag{34}$$

Applying the above to the resolvent functions

$$R(\lambda - i\epsilon) - R(\lambda + i\epsilon) = (\lambda - A - i\epsilon)^{-1} - (\lambda - A + i\epsilon)^{-1} \tag{35}$$

then we have by Riesz-Markov for this measure $d\mu_u$ and A that

$$\left(\frac{1}{2\pi i}(R(\lambda - i\epsilon) - R(\lambda + i\epsilon))u, u\right) = \int_{\sigma(A)} (\lambda - t - i\epsilon)^{-1} - (\lambda - t + i\epsilon)^{-1} d\mu_u(t) \tag{36}$$

and extending $\mu_u(t)$ trivially outside of its support on $\sigma(A)$ to a regular measure $\tilde{\mu}_u(t)$ we can rewrite this (with slight abuse of notation) as

$$\left(\frac{1}{2\pi i}(R(\lambda - i\epsilon) - R(\lambda + i\epsilon))u, u\right) = \frac{1}{2\pi i}\int_{\mathbb{R}} (\lambda - t - i\epsilon)^{-1} - (\lambda - t + i\epsilon)^{-1} d\tilde{\mu}_u(t). \tag{37}$$

Thus we note that that for $\epsilon > 0$ arbitrary we have that

$$\int_a^b \left(\frac{1}{2\pi i}(R(\lambda - i\epsilon) - R(\lambda + i\epsilon))u, u\right) f(\lambda) d\lambda = \int_a^b \frac{1}{2\pi i}\int_{\mathbb{R}} (\lambda - t - i\epsilon)^{-1} - (\lambda - t + i\epsilon)^{-1} d\tilde{\mu}_u(t) f(\lambda) d\lambda. \tag{38}$$

Now we note that we can rewrite the integrand as

$$\frac{1}{2\pi i}((\lambda - t - i\epsilon)^{-1} - (\lambda - t + i\epsilon)^{-1}) f(\lambda) = \frac{1}{\pi}\frac{\epsilon}{\epsilon^2 + (\lambda - t)^2} f(\lambda) \tag{39}$$

Therefore for $\epsilon > 0$ fixed we know that

$$\int_a^b \left|\frac{1}{\pi}\frac{\epsilon}{\epsilon^2 + (\lambda - t)^2} f(\lambda)\right| d\lambda \le C \int_a^b \frac{1}{\pi}\frac{\epsilon}{\epsilon^2 + (\lambda - t)^2} d\lambda \tag{40}$$

$$= \frac{1}{\pi}\left(\arctan\left(\frac{b-t}{\epsilon}\right) - \arctan\left(\frac{a-t}{\epsilon}\right)\right) \le 1, \tag{41}$$

where we set $C := \sup_{\lambda\in(a,b)} |f(\lambda)|$.

$$\int_a^b \int_{\mathbb{R}} \frac{1}{2\pi i}((\lambda - t - i\epsilon)^{-1} - (\lambda - t + i\epsilon)^{-1})f(\lambda)d\tilde{\mu}_u(t)d\lambda \tag{42}$$

$$= \int_{\mathbb{R}} \int_a^b \frac{1}{2\pi i}((\lambda - t - i\epsilon)^{-1} - (\lambda - t + i\epsilon)^{-1})f(\lambda)d\tilde{\mu}_u(t)d\lambda. \tag{43}$$

$$= \int_{\mathbb{R}} \int_a^b \frac{1}{\pi}\frac{\epsilon}{\epsilon^2 + (\lambda - t)^2} f(\lambda)d\lambda d\tilde{\mu}_u(t) \tag{44}$$

Now we still want to take a limit $\epsilon \searrow 0$ inside the first integral. For this we apply dominated convergence. As $f \in C_c^0(a,b)$ and (a,b) is a bounded interval we can choose a $C > 0$ such that $|f(\lambda)| < \frac{C}{1+\lambda^2}$ for all $\lambda \in (a,b)$. Let us set $g_\epsilon(t)$ as:

$$g_\epsilon(t) = \frac{1}{\pi}\int_a^b \frac{\epsilon}{\epsilon^2 + (\lambda - t)^2} f(\lambda)d\lambda, \tag{45}$$

then one can calculate that

$$|g_\epsilon(t)| \le \frac{1}{\pi}\int_{\mathbb{R}} \frac{C}{1+\lambda^2}\frac{\epsilon}{\epsilon^2 + (\lambda - t)^2} d\lambda = \frac{(1+\epsilon)C}{(1+\epsilon^2+t^2}, \tag{46}$$

hence $g_\epsilon(t)$ is uniformily bounded for all $\epsilon > 0$. To summarize, putting all of the above together, up until now we have the following equalities

$$\lim_{\epsilon\searrow 0} \frac{1}{2\pi i}\int_a^b ((R(\lambda - i\epsilon) - R(\lambda + i\epsilon))u, u)\, f(\lambda)d\lambda = \lim_{\epsilon\searrow 0}\int_a^b \int_{\mathbb{R}} \frac{1}{\pi}\frac{\epsilon}{\epsilon^2 + (\lambda - t)^2} f(\lambda)\; d\tilde{\mu}_u(t)d\lambda \tag{47}$$

$$= \lim_{\epsilon\searrow 0}\int_{\mathbb{R}}\int_a^b \frac{1}{\pi}\frac{\epsilon}{\epsilon^2 + (\lambda - t)^2} f(\lambda)\; d\lambda d\tilde{\mu}_u(t) \tag{48}$$

$$= \int_{\mathbb{R}}\lim_{\epsilon\searrow 0}\int_a^b \frac{1}{\pi}\frac{\epsilon}{\epsilon^2 + (\lambda - t)^2} f(\lambda)\; d\lambda d\tilde{\mu}_u(t), \tag{49}$$

where the main steps were Fubini and dominated convergence. We are done when we can show that

$$\int_a^b \frac{1}{\pi}\frac{\epsilon}{\epsilon^2 + (\lambda - t)^2} f(\lambda)\; d\lambda = f(t). \tag{50}$$

Let us define $\psi_\epsilon(\lambda - t)$ as

$$\psi_\epsilon(\lambda - t) = \frac{1}{\pi}\frac{\epsilon}{\epsilon^2 + (\lambda - t)^2}, \tag{51}$$

we claim that it is an approximation of the Dirac distribution supported at $\{t\}$ i.e. $\psi_\epsilon(\lambda - t)$ satisfies $\psi_\epsilon(\cdot - t) \to \delta(\cdot - t)$ in the sense of distributions:

$$\lim_{\epsilon \searrow 0} \int_{\mathbb{R}} \psi_\epsilon(\lambda - t) f(t) d\lambda = f(t) \text{ for all } f \in C^0(\mathbb{R}) \tag{52}$$

and therefore

$$\lim_{\epsilon \searrow 0} \int_a^b \psi_\epsilon(\lambda - t) f(\lambda) d\lambda = f(t) \text{ for all } f \in C_c^0(a, b). \tag{53}$$

WLOG we will prove that this holds in $t = 0$, i.e. for

$$\psi_\epsilon(\lambda) = \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + \lambda^2} \tag{54}$$

It will follow from the lemma below, if we check three things: 1) $\psi_\epsilon(\lambda) \geq 0$ for all $\epsilon > 0$, 2) $\int_{\mathbb{R}} \psi_\epsilon(\lambda)$ and 3) for $\delta > 0$ arbitrary, we have

$$\lim_{\epsilon \searrow 0} \int_{\mathbb{R} \setminus [-\delta, \delta]} \psi_\epsilon(\lambda) = 0. \tag{55}$$

It is trivial that $\psi_\epsilon(\lambda) \geq 0$ for all $\epsilon > 0$. The second identity follows quickly, we have

$$\int_{\mathbb{R}} \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + \lambda^2} d\lambda = \frac{1}{\pi} \lim_{\lambda \to \infty} (\arctan(\lambda/\epsilon) - \arctan(-\lambda/\epsilon)) \tag{56}$$

$$= 1, \tag{57}$$

by changing variables to $x = \lambda/\epsilon$.
Finally the third requirement follows from the fact that

$$\lim_{\epsilon \searrow 0} \int_{\mathbb{R} \setminus [-\delta, \delta]} \psi_\epsilon(\lambda) d\lambda = \lim_{\epsilon \searrow 0} \frac{1}{\pi} ((\frac{\pi}{2} - \underbrace{\arctan(\frac{\delta}{\epsilon})}_{\to \frac{\pi}{2}}) - (\underbrace{(\arctan(\frac{\delta}{\epsilon})}_{\to \frac{\pi}{2}} - \frac{\pi}{2}) \tag{58}$$

$$= 0. \tag{59}$$

combined with the lemma below this proves that

$$\int_a^b \psi_\epsilon(\lambda - t) f(\lambda) d\lambda = f(t). \tag{60}$$

We conclude now that we have proven

$$\lim_{\epsilon \searrow 0} \frac{1}{2\pi i} \int_a^b \Big(\big(R(\lambda - i\epsilon) - R(\lambda + i\epsilon) \big) u, u \Big) f(\lambda) \, \mathrm{d}\lambda = \int_{\mathbb{R}} \lim_{\epsilon \searrow 0} \int_a^b \psi_\epsilon(t - \lambda) f(\lambda) d\lambda d\tilde{\mu}_u(t) \tag{61}$$

$$= \int_{\mathbb{R}} f(t) d\tilde{\mu}_u(t) \tag{62}$$

$$= \int \sigma(A) f(t) d\mu(t) \tag{63}$$

$$= (f(A)u, u) \tag{64}$$

as desired. We note that the weak $*$-convergence on the dual space of $C_c^0(\mathbb{R})$ that is being mentioned here is that of regular Borel measures on $\mathbb{R}$, often denote as the set $\mathcal{M}(\mathbb{R})$. We identify regular measures $\mu \in \mathcal{M}(\mathbb{R})$ with functionals $L_\mu : C_c^0(X) \to \mathbb{C}$ as $L_\mu(f) = \int_{\mathbb{R}} f d\mu$. We recall that the weak-$*$ topology on $C_c^0(\mathbb{R})$ is given by the smallest topology that makes evaluation functions of these functionals continuous, i.e. the functionals $\mathcal{M}(X)^* \ni \hat{f} : \mathcal{M}(X) \to \mathbb{C}$ that for $f \in C_c^0(\mathbb{R})$ are defined by

$$\hat{f}(\mu) = \mu(f) = \int_{\mathbb{R}} f d\mu. \tag{65}$$

Clearly the spectral measure of $u \in H$, μ_u is a regular measure on $\mathbb{R}$. As $f \in C_c^0(\mathbb{R})$ is arbitrary and we can also extend the integration bounds trivially and get

$$\lim_{\epsilon \searrow 0} \frac{1}{2\pi i} \int_{\mathbb{R}} \Big(\big(R(\lambda - i\epsilon) - R(\lambda + i\epsilon) \big) u, u \Big) f(\lambda) \, \mathrm{d}\lambda = \int_{\mathbb{R}} f(t) d\tilde{\mu}(t) \tag{66}$$

from where we conclude that $\hat{f}\left(\frac{1}{2\pi i}((R(\cdot - i\epsilon) - R(\cdot + i\epsilon))u, u)\mathcal{L}\right) \to \hat{f}\,(d\tilde{\mu}_u)$ as $\epsilon \searrow 0$, where $d\mathcal{L}$ is the standard Lebesque measure, for any evaluation function $\hat{f}$ defined by $f \in C_c^0(a, b)$. This is precisely the definition of a limit in the weak-$*$ topology on $\mathcal{M}(\mathbb{R})$.

Lemma. *We say that the family $(\eta_\epsilon)_{\epsilon \geq 0} \subset L^1(\mathbb{R})$ is called an approximating sequence of the δ-distribution the following three statements hold:*

$$i)\ \eta_\epsilon \geq 0 \ \ \forall \ \ \epsilon \geq 0 \tag{67}$$

$$ii)\ \int_{\mathbb{R}} \eta_\epsilon dx = 1 \ \ \forall \ \ \epsilon \geq 0 \tag{68}$$

$$iii)\ \lim_{\epsilon \to 0} \int_{\mathbb{R} \setminus [-\delta, \delta]} \eta_\epsilon dx = 0 \ \ \forall \ \ \delta > 0 \ \ \textit{fixed.} \tag{69}$$

Then we have that $\eta_\epsilon \to \delta(x)$ as $\epsilon \searrow 0$, i.e. we have

$$\lim_{\epsilon \searrow 0} \int_{\mathbb{R}} \phi(x) \eta_\epsilon(x) \, dx = \phi(0) \textit{ for all } \phi \in C_c^0(\mathbb{R}). \tag{70}$$

Proof. Let $\phi \in C_c^0(\mathbb{R})$ and $\tilde{\epsilon} > 0$. Then we have that

$$\left| \int_{\mathbb{R}} \phi \cdot \eta_\epsilon \, dx - \phi(0) \right| = \left| \int_{\mathbb{R}} \phi \cdot \eta_\epsilon dx - \phi(0) \underbrace{\int_{\mathbb{R}} \eta_\epsilon dx}_{=1} \right| \tag{71}$$

$$= \left| \int_{\mathbb{R}} (\phi(x) - \phi(0)) \eta_\epsilon dx \right| \tag{72}$$

$$\leq \int_{[-\delta, \delta]} |\phi(x) - \phi(0))| \, \eta_\epsilon(x) dx + \int_{\mathbb{R} \setminus [-\delta, \delta]} |\phi(x) - \phi(0))| \, \eta_\epsilon dx, \tag{73}$$

where in we used (67) in the second line. Thus if we choose $\delta > 0$ in such a way that $|\phi(x) - \phi(0)| < \tilde{\epsilon}/2$ (by continuity of ϕ), and $\epsilon(\delta) > 0$ such that $\tilde{\epsilon} \to 0$, for

$$\int_{\mathbb{R}\setminus[-\delta,\delta]} \eta_\epsilon dx < \min\{\tilde{\epsilon}/2, 1\}, \tag{74}$$

we have that

$$\int_{[-\delta,\delta]} |\phi(x) - \phi(0))|\, \eta_\epsilon(x) dx \leq \frac{\tilde{\epsilon}}{2} \underbrace{\int_{\mathbb{R}} \eta_\epsilon(x) dx}_{=1} = \frac{\tilde{\epsilon}}{2} \tag{75}$$

and

$$\int_{\mathbb{R}\setminus[-\delta,\delta]} |\phi(x) - \phi(0))|\, \eta_\epsilon dx < \frac{\tilde{\epsilon}}{2} \min\{1, \frac{\tilde{\epsilon}}{2}\} \leq \frac{\tilde{\epsilon}}{2}, \tag{76}$$

and we are done.

□

Remark. Hopefully the steps in the above and the approximation of the delta distributions are something that might have been treated in a previous lecture on analysis (they are basically a family of Poisson kernels). The formula that was proven in (61) is also known as *Stone's formula.*

3.4. Diagonalization of $\frac{\mathrm{d}}{\mathrm{d}t}$.

This is a simple applications of the concepts we used in the first exercise of this sheet. We know from exercise 1.1 that A is self-adjoint. The obvious unitary transformation (isometric isomorphism) would then be the Fourier transform applied to $D(A) \cong H^1(\mathbb{S}^1)$. Clearly $\mathcal{F} : D(A) \to \ell^2(\mathbb{Z})$ and we have for a given $(a_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$

$$\mathcal{F} \circ i\frac{d}{dt} \circ \mathcal{F}^{-1}(a_n)_{n\in\mathbb{Z}} = (-na_n)_{n\in\mathbb{Z}}. \tag{77}$$

Clearly the space we are looking for is therefore $M = (\ell^2(\mathbb{Z}), \mu_{\mathbb{Z}})$ where $\mu_{\mathbb{Z}} : \mathbb{Z} \to [0, \infty)$ is the discrete counting measure. The function $g : \mathbb{Z} \to \mathbb{R}$ is clearly given by

$$g(n) = -n. \tag{78}$$

In this way $T_g : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is simply given by $T_g(a_n)_{n\in\mathbb{Z}} = (-na_n)_{n\in\mathbb{Z}}$.