4.1. Explicit form of a unitary transformation of a self-adjoint operator.

(a) This entire exercise can be very nicely solved in the same vein as exercises 3.1 and 3.4 of the previous sheet when we note that $D(A) \cong H^1(\mathbb{S}^1)$. We then know that the Fourier transform $\mathcal{F} : H^1(\mathbb{S}^1) \to h^1(\mathbb{S}^1)$ is the unitary transformation that diagonalizes $i\frac{d}{dx}$, we have for $(a_n)_{n\in\mathbb{Z}} \in h^1(\mathbb{Z})$, as from exercise 3.4 we have

$$\mathcal{F} \circ i\frac{d}{dx} \circ \mathcal{F}^{-1}(a_n)_{n\in\mathbb{Z}} = (-na_n)_{n\in\mathbb{Z}}, \tag{1}$$

hence in the language of the spectral theorem for unbounded self-adjoint functions the measurable function $g : \mathbb{Z} \to \mathbb{R}$ that we are after is given by $g(n) = -n$. Now the strongly continuous operator family e^{itA} for $t \in \mathbb{R}$ is easily expressed on the $\ell^2(\mathbb{Z})$ as e^{-itn} and we get the actual operator by conjugating with $\mathcal{F}$:

$$e^{itA}f(x) = (\mathcal{F}^{-1} \circ e^{itg(n)} \circ \mathcal{F})f(x).$$

Using that $f(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}$ we can then compute directly

$$\begin{aligned} e^{itA}f(x) = \mathcal{F}^{-1} \circ (e^{-itn}a_n)_{m\in\mathbb{Z}} &= \sum_{n\in\mathbb{Z}} a_n e^{-itn}e^{inx} \\ &= \sum_{n\in\mathbb{Z}} a_n e^{in(x-t)} = f(x-t). \end{aligned}$$

We conclude that e^{itA} is translation by t.

(b) With our knowledge of the previous exercise, this is also proven rather easily. If we define the unitary operator $U \in C^1((0,1), L(L^2(0,1))$ via

$$U(x)f(x) = e^{-i\alpha x}f(x), \tag{2}$$

then if $f \in D(A) = \{f \in H^1(0,1) | f(0) = e^{i\alpha}f(1)\}$ we have

$$U(1)f(1) = e^{-i\alpha}f(1) = e^{-i\alpha}e^{i\alpha}f(0) = U(0)f(0),$$

i.e. $U((0,1)\times D(A)) = H^1(\mathbb{S}^1)$. As U is also unitary, let us first analyze $U(x)\circ i\frac{d}{dx}\circ U(x)^{-1} : H^1(\mathbb{S}^1) \hookrightarrow D(A) \to L^2(\mathbb{S}^1)$, where $i\frac{d}{dx} : D(A) \to L^2(\mathbb{S}^1)$. We emphasize here that $i\frac{d}{dx} : D(A) \to L^2(0,1)$ is *not* the same operator as $i\frac{d}{dx} : H^1(\mathbb{S}^1) \to L^2(0,1)$ in the exercise above, i.e. it will be unitarily equivalent to a *different* operator $\tilde{A} : H^1(\mathbb{S}^1) \to L^2(\mathbb{S}^1)$. We calculate for $\psi \in H^1(\mathbb{S}^1)$ using the product rule

$$\begin{aligned} (U(x) \circ i\frac{d}{dx} \circ U(x)^{-1})\psi(x) &= e^{-i\alpha x}i\frac{d}{dx}e^{i\alpha x}\psi(x) \\ &= i\frac{d}{dx}\psi(x) - \alpha\psi(x). \end{aligned}$$

i.e. $i\frac{d}{dx}$ on $D(A)$ is unitarily equivalent to $i\frac{d}{dx} - \alpha$ on $H^1(\mathbb{S}^1)$. Obviously, the Fourier transform diagonalizes this operator again, and we get for $(a_n)_{n\in\mathbb{Z}}$ that

$$\mathcal{F}(U(x) \circ i\frac{d}{dx} \circ U(x)^{-1})\mathcal{F}^{-1}(a_n)_{n\in\mathbb{Z}} = ((-n-\alpha)a_n)_{n\in\mathbb{Z}}, \tag{3}$$

i.e. the multiplication operator $g : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ given by $g(n) = -n - \alpha$ Thus exponentiating we get that

$$\mathcal{F} \circ (U(x) \circ e^{itA} \circ U(x)^{-1}) \circ \mathcal{F}^{-1}(a_n)_{n\in\mathbb{Z}} = (e^{-i(n-\alpha)t}a_n)_{n\in\mathbb{Z}}, \tag{4}$$

and hence for $f \in D(A)$ we calculate

$$\begin{aligned} U(x) \circ \mathcal{F} \circ e^{i(n-\alpha)t} \circ \mathcal{F}^{-1} \circ U(x)^{-1} f(x) &= U(x)^{-1} \circ \mathcal{F}^{-1} \circ e^{-i(n-\alpha)t} \circ \mathcal{F} \circ U(x) f(x) \\ &= U(x)^{-1} \circ \mathcal{F}^{-1} e^{-i(n-\alpha)t} \circ \mathcal{F}\psi(x) \\ &= U(x)^{-1} \circ \mathcal{F}^{-1}(e^{-i(n-\alpha)t}a_n)_{n\in\mathbb{Z}} \\ &= U(x)^{-1} \circ \mathcal{F}^{-1}(e^{-i(n-\alpha)t}a_n)_{n\in\mathbb{Z}} \\ &= U(x)^{-1}\left(\sum_{n\in\mathbb{Z}} a_n e^{in(x-t)-i\alpha t}\right) \\ &= U(x)^{-1}\psi(x-t)e^{-i\alpha t} \\ &= f(x-t)e^{-i\alpha t}, \end{aligned}$$

where we use the identification $H^1(\mathbb{S}^1) \ni \psi(x) := U(x)f(x) \in H^1(\mathbb{S}^1)$ for $f \in D(A)$ and the Fourier decomposition

$$\psi(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx}. \tag{5}$$

[1]

4.2. Cauchy's formula for the spectrum of self-adjoint operators.

To start of let us analyze the equality that we are trying to prove. On the l.h.s. we have $f(A)$ as it is given in the context of the continuous and Borel functional calculus that we know quite well at this point. The equation on the right hand side is the definition of $f(A)$ through the so-called *holomorphic* functional calculus also known as the *Riesz-Dunford* functional calculus. This exercise basically asks us to prove that the definition of both functional calculi agree when f is holomorphic and A is bounded, but before we can

[1]Actually one would expect a factor $e^{2\pi inx}$ if we let $x \in (0,1)$ but we can always rescale this of course.

prove anything we first need to make sure that the r.h.s even makes sense. I.e. that the map for fixed $A \in L(H)$ $\Phi_{\mathcal{H}} : \mathcal{H}(\sigma(A)) \to L(H)$ given by [2]

$$\Phi(f) = f_{\mathcal{H}}(A) := \frac{1}{2\pi i} \oint_{\gamma} f(z)(z - A)^{-1} dz \tag{6}$$

is well defined. Well-definedness in this case includes both making sure that the vector valued integral on the r.h.s is well-defined. Then we will show that $f_{\mathcal{H}}(A) = f(A)$ whilst simultaneously showing that $f_{\mathcal{H}}$ would not have changed if we had chosen another homologous path $\tilde{\gamma}$. We will use a subscript $\mathcal{H}$ to distinguish the holomorphic functional calculus from the Borel functional calculus as at this point it is not a priori evident that they indeed coincide.

Vector valued integrals.

First, we need to make sure that the right hand side is well defined. Let us first observe that as $A \in L(H)$ is bounded, $\sigma(A)$ is compact, so lets assume that $\sigma(A) \subseteq [a, b]$. So for any open U that contains $\sigma(A)$ we can select a piecewise C^1 path that winds around $\sigma(A)$ once and indeed the rectangle suggested would do the trick if we make ϵ small enough. It will however be convinient for the exercise to even assume that we C^∞ curve γ. Second, we need to make sense of what the integral on the r.h.s. actually is in this case, and in what follows we leave out a lot of the details. We observe that the function

$$\mathbb{C} \supset \gamma \ni z \mapsto \frac{1}{2\pi i} f(z)(z - A)^{-1} \in L(H) \tag{7}$$

defines a function $\gamma \to L(H)$ as the path γ lies outside of $\sigma(A)$ by assumption. The integral of this operator valued (we still call this vector valued) function is defined in much the same way as one knows from the Lebesque integral and are called *Bochner* integrals. We will record a number of definitions and theorems around Bochner integrals below.

Definition. *Let (X, Σ, μ) be a measure space and B a Banach space then simple functions*

$$s : X \to B$$

are defined as

$$s(x) = \sum_{i=1}^{n} b_i \chi_{E_i} \quad \textit{where} \;\; b_i \in B \;\; \textit{and} \;\; E_i \in \Sigma. \tag{8}$$

for some $n \in \mathbb{N}_0$

[2]Here we read $\mathcal{H}(\sigma(A))$ as the germ of holomorophic functions on an open neighbourhood of $\sigma(A)$

Definition. *Let (X, Σ, μ) be a measure space and B a Banach space. We say that a function $f : X \to B$ is Bochner measurable (strongly measurable), if there exists a sequence $\{s_k\}$ of simple functions $s_k : X \to B$ such that*

$$f(x) = \lim_{k\to\infty} s_k(x) \text{ for } x \in X,\ \mu - \text{almost everywhere.} \tag{9}$$

We denote the stronly measurable functions $f : X \to B$ with $\mathcal{M}(X, \Sigma, \mu; B)$. One can also define weakly measurable functions as functions f for which for all $\phi \in X^*$, the map

$$X \ni x \mapsto \phi(f(x)) \in \mathbb{C} \tag{10}$$

is measurable.

Definition. *(Bochner integral, Riemannian version) Consider $[a, b] \subset \mathbb{R}$ with the standard Lebesque measure (we will omit notation for the σ-algebra etc.), and let B be a Banach space. Let $f[a, b] \to B$ be strongly measurable, and let $T = \{a = t_1 < t_2, ..., t_n = b$ be a partition of T. We define the mesh of T as*

$$||T|| = \sup_{1\le i\le T} |t_{i+1} - t_i| \tag{11}$$

and a set of intermediate points ξ_T subordinate to T as $\xi_T = \{\xi_i \in [0, 1]|\xi_i \in [t_i, t_{i+1}]\}$. For these quantities we define the Riemann sum

$$S(f, T, \xi_T) := \sum_{i=0}^{n-1} (t_{i+1} - t_i) f(\xi_i). \tag{12}$$

We say that f is Riemann integrable if there exists some $b \in B$ such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every partition T and intermediate points ξ_T satisfying $||T|| < \delta$ we have

$$||S(f, T, \xi_T) - b||_B < \epsilon. \tag{13}$$

In this case we write

$$\int_a^b f(t)dt = b. \tag{14}$$

The following theorems should come as no suprise.

Theorem. *Let $f, g : [a, b] \to B$ be Riemann integrable. Then for $\mu, \lambda \in \mathbb{C}$ we have*

$$\int_a^b \lambda f(x) + g(x)dx = \lambda \int_a^b f(x)dx + \int_a^b g(x)dx. \tag{15}$$

Moreover, if $||f(x)||_B : [a, b] \to \mathbb{R}_+$ is Riemann integrable iff $f : [a, b] \to B$ is Riemann integrable, and we have

$$\left|\left|\int_a^b f(x)dx\right|\right|_B \le \int_a^b ||f(x)||_B dx \tag{16}$$

Moreover, if $f : [a, b] \to B$ is continuous, it is Riemann integrable.

Proof. Exercise for the reader. □

Now we have to make sense of the taking a line integral $\gamma \subset \mathbb{C}$ where we will assume γ to simply be smooth.

Definition. *Let $\gamma : [0,1] \to \mathbb{C}$ be a closed C^∞ curve and let $\psi \in \mathcal{M}(\mathbb{C}\sigma(A), \Sigma, \mu; B)$ be strongly measurable. We define the integral of ψ over γ as*

$$\oint_\gamma \psi(z)dz = \int_0^1 \psi(\gamma(t))\gamma'(t)dt, \tag{17}$$

where the integral in the r.h.s. is understood to be the vector valued integral Riemann integral of the function $(\psi \circ \gamma)\gamma'$.[3]

Now as $\gamma \cap \sigma(A) = \emptyset$, we know that $f(z)(z-A)^{-1}$ is analytic (see also exercise 3) on γ and so certainly $f(\gamma(t))(\gamma(t)-A)^{-1}\gamma'(t) : [0,1] \to L(H)$ continuous. Most certainly then, the integral

$$\oint_\gamma f(z)(z-A)^{-1}dz = \int_0^1 f(\gamma(t))(\gamma(t)-A)^{-1}\gamma'(t)dt \in L(H). \tag{18}$$

is well defined and yields an operator $b \in L(H)$.

We also note that we have the following theorem.

Theorem. *Let $\psi : [a,b] \to L(H)$ be a continuous function and let $u, v \in H$ arbitrary. Then we have*

$$\langle \int_0^1 \psi(t)\ dtu, v\rangle_H = \int_0^1 \langle \psi(t)u, v\rangle_H dt \tag{19}$$

Proof. Let T be an arbitrary mesh for $[0,1]$ and ξ_T its set of intermediate points. We then see that this arbitrary Riemann sum $S(\psi, T, \xi_T)$ we have

$$\begin{aligned}\langle S(\psi, T, \xi_T)u, v\rangle_H &= \left\langle \sum_{i=1}^n \psi(\xi_i)(t_{i+1}-t_i)u, v \right\rangle_H \\ &= \sum_{i=1}^n \langle \psi(\xi_i)u, v\rangle_H (t_{i+1}-t_i) \\ &= S(\langle \psi(t)u, v\rangle_H, T, \xi_T).\end{aligned}$$

[3]Note that $\gamma'(t) \in \mathbb{C}$ for all $t \in [0,1]$ so it is a scalar!

Notice as a quick reality check that $[0,1] \ni t \mapsto \langle \psi(t)u, v\rangle_H \in \mathbb{C}$ is simply a continuous function and thus also integrable over $[0,1]$ as

$$\int_0^1 \langle \psi(t)u, v\rangle_H dt \leq \max_{t\in[0,1]} ||\psi(t)||_{L(H)}||u||_H||v||_H.$$

Therefore taking limits of partitions with mesh $||T|| \to 0$ and corresponding ξ_T we get

$$\langle \int_0^1 \psi(t)\ dtu, v\rangle_H = \int_0^1 \langle \psi(t)u, v\rangle_H dt$$

as required. □

From this and our knowledge of $f(z)(z-A)^{-1}$ we immediately get the following identity for $u, v \in H$ arbitrary

$$\left\langle \oint_\gamma f(z)(z-A)^{-1}dz\ u, v\right\rangle_H = \oint_\gamma \langle\langle f(z)(z-A)^{-1}dz\ u, v\rangle_H dz. \tag{20}$$

With this last argument done, we are now set up to prove equality to the Borel functional calculus, and show that the holomorphic functional calculus is independent of the path γ chosen. We recall that through the Borel functional calculus we can define

$$\langle f(A)u, u\rangle_H = \int_{\mathbb{R}} f(\lambda)d\mu_u(\lambda),$$

where μ_u is supported on $\sigma(A)$. Notice in particiular that for all $u \in H$ we have that μ_u is a finite measure as

$$\infty > ||u||_H^2 = \int_{\mathbb{R}} 1 d\mu_u = \mu_u(\mathbb{R}).$$

As always, for the bilinear $\langle\cdot,\cdot\rangle_H$ know that the polarization identity

$$2\langle Au, v\rangle_H = \langle T(u+v), u+v\rangle_H + i\langle T(u+iv), u+iv\rangle_H - (1+i)\langle Tu, u\rangle_H - (1+i)\langle Tv, v\rangle_H \tag{21}$$

must hold. This, combined with the fact that μ_u and μ_v are finite, we can define the *complex* measure

$$\mu_{u,v} = \frac{1}{2}\left(\mu_{u+v} + i\mu_{u+iv} - (1+i)\mu_u - (1+i)\mu_v\right). \tag{22}$$

It is easy to check that for this measure it holds that for A and $u, v \in H$ fixed,

$$\langle Au, v\rangle_H = \int_{\mathbb{R}} f(\lambda)d\mu_{u,v}(\lambda), \tag{23}$$

where $\mu_{u,v}$ is again supported on $\sigma(A)$. We will use these measures $\mu_{u,v}$ as it is slightly more convenient in what follows. For the punchline of this entire exercise we replace the standard Lebesque measure $d\mu(z)$ on $\mathbb{C}$ with the dz so as not to confuse it with the Borel spectral measure $d\mu_{u,v}$ we find

$$\begin{aligned}
\langle f_{\mathcal{H}}(A)u, v\rangle_H &= \left\langle \frac{1}{2\pi i}\oint_\gamma f(z)(z-A)^{-1}\,dz\; u, v\right\rangle_H \\
&= \frac{1}{2\pi i}\oint_\gamma \left\langle f(z)(z-A)^{-1}\;\; u, v\right\rangle_H dz \\
&= \frac{1}{2\pi i}\oint_\gamma \int_{\sigma(A)} \frac{f(z)}{z-\lambda}\,d\mu_{u,v}(\lambda)\,d(z) \\
&= \frac{1}{2\pi i}\int_0^1 \int_{\sigma(A)} \frac{f(\gamma(t))}{\gamma(t)-\lambda}\gamma'(t)\,d\mu_{u,v}(\lambda)\,dt \\
&= \int_{\sigma(A)} \frac{1}{2\pi i}\int_0^1 \frac{f(\gamma(t))}{\gamma(t)-\lambda}\gamma'(t)\,dt\,d\mu_{u,v}(\lambda) \\
&= \int_{\sigma(A)} \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z-\lambda}\,dz\,d\mu_{u,v}(\lambda) \\
&= \int_{\sigma(A)} f(\lambda)\,d\mu_{u,v}(\lambda) \\
&= \langle f(A)u, v\rangle_H.
\end{aligned}$$

In the second line we used the interchanging of the contour integral and the scalar product on H for vector valued functions, in the third line we plugged in the Borel functional calculus, used Fubini to swap, and finally we were able to rewrite this as a standard contour integral of a holomorphic function on $\mathbb{C}\backslash\sigma(A)$ in (**??**) and finally apply the Cauchy integral formula in this ordinary setting. [4]) to completely rewrite this into the standard expression for the Borel functional calculus. As we now know $\langle f_{\mathcal{H}}(A)u, v\rangle_H = \langle f(A)u, v\rangle_H$ for arbitrary $u, v \in H$ we conclude with the Riesz representation theorem for Hilbert spaces that $f(A) = f_{\mathcal{H}}(A)$. Finally to edge out well-definedness of the holomorphic functional calculus: let $\tilde{\gamma}$ be another path winding around $\sigma(A)$ once, and assume it to be homologous γ. If we were to define $\tilde{f}_{\mathcal{H}}(A)$ as

$$\tilde{f}_{\mathcal{H}}(A) = \frac{1}{2\pi i}\oint_{\tilde{\gamma}} f(z)(z-A)^{-1}dz \tag{24}$$

[4] Although the Cauchy interal formula can be proven for vector valued integrals of analytic functions as well, the nice thing about our current setup is that by swapping integrals around we can rewrite the integral to an integral for which we already know the Cauchy integral formula to hold.

we note that we can run the above computations again up until (**??**) and find

$$\langle \tilde{f}_{\mathcal{H}}(A)u, v\rangle_H = \int_{\sigma(A)} \frac{1}{2\pi i} \oint_{\tilde{\gamma}} \frac{f(z)}{z-\lambda}\, dz\, d\mu_{u,v}(\lambda) \tag{25}$$

$$= \int_{\sigma(A)} \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z-\lambda}\, dz d\mu_{u,v}(\lambda) \tag{26}$$

$$= \langle f_{\mathcal{H}}(A)u, v\rangle_H, \tag{27}$$

where we shift the contour from γ to $\tilde{\gamma}$ for an ordinary holomorphic function![5] Again with Riesz we conclude that the holomorphic calculus is well-defined, and independent of the homologous paths γ and $\tilde{\gamma}$ chosen.

Remark. There is a nicer version of the Bochner integrals which is reminiscent of Lebesque integration. But in this formalism the commuting identity (20) is harder to prove as you work with simple functions on $[0, 1]$ that do not translate back nicely to a contour integral. For people more interested in the Riesz-Dunford holomorphic calculus, or in Bochner integrals integrals in general I recommend reading the works of Rudin, Lax or Yosida (all called Functional Analysis) or the original paper by Schwartz and Dunford, called Linear Operators: Part I. *Vector Measures* by Diestel and Uhl also has a good introduction to vector valued integrals.

4.3. Analicity of the resolvent operator

(a) We know that by definition that if $z \in \rho(A)$, it holds that

$$R_z(z-A) = Id = (z-A)R_z,$$

therefore consider the quantity

$$R_w \circ ((w-A) - (z-A)) \circ R_z = R_z - R_w. \tag{28}$$

where we note that the composition of the l.h.s is well-defined as $R_z, R_w : H \to D(A)$ are continuous bijections that map onto the domain of A. Now noting that the l.h.s reduces to

$$R_w((w-A) - (z-A))R_z = (w-z)R_wR_z, \tag{29}$$

gives us the required identity.

[5] In fact we could have concluded that $f_{\mathcal{H}}(A)$ is independent of the homologous paths γ or $\tilde{\gamma}$ by the fact that it is equal to $f(A)$ yielded by the Borel functional calculus. The latter is of course not defined using any contour at all...

(b) Let $z_0 \in \rho(A)$ arbitrary. Choose $z \in \mathbb{C}$ such that $|z - z_0| \le ||R_{z_0}||^{-1}$, where $||R_{z_0}||$ is the operator norm of $R_{z_0} = (z_0 - A)^{-1}$ and is bounded by assumption. Clearly then as

$$\frac{|z - z_0|}{||R_{z_0}||} < 1, \tag{30}$$

we have that for the operator $T : D(A) \to H$ defined by

$$T := \sum_{n=0}^{\infty} (-1)^n (z - z_0)^n ((z_0 - A)^{-1})^n \tag{31}$$

that

$$||T|| \le ||\sum_{n=0}^{\infty} (-1)^n (z - z_0)^n ((z_0 - A)^{-1})^n|| \le \sum_{n=0}^{\infty} \frac{|z_0 - z|^n}{||R_{z_0}||} < \infty \tag{32}$$

is bounded, as the r.h.s is absolutely convergent.
Now we perform a simple Neumann-series argument yields us that

$$\begin{aligned}(z - A) \circ T &= ((z - z_0)Id_H + (z_0 - A)) \sum_{n=0}^{\infty} (-1)^n (z - z_0)^n ((z_0 - A)^{-1})^n \\ &= \cancel{(z - z_0)Id_H} + (z_0 - A) - \cancel{(z - z_0)^2((z_0 - A)^{-1}} - \cancel{(z - z_0)Id_H} \\ &+ (z - z_0)^3((z_0 - A)^{-1})^2 + \cancel{(z - z_0)^2(z_0 - A)^{-1}} + ... \\ &= (z_0 - A).\end{aligned}$$

Now $(z_0 - A)$ is invertible by assumption (as we assumed $z_0 \in \rho(A)$, therefore we conclude that

$$T \circ (z_0 - A)^{-1} := \sum_{n=0}^{\infty} (-1)^n (z - z_0)^n ((z_0 - A)^{-1})^{n+1} \tag{33}$$

is the right inverse of $(z - A)$ and similarly one can show that it is the left inverse as well, i.e. $T \circ (z_0 - A)^{-1} =$. We conclude that $(z - A)$ is invertible, therefore $z \in \rho(A)$. With this we have proved that the ball with radius $||R_{z_0}||^{-1}$ is contained in $\rho(A)$ for arbitrary $z_0 \in \rho(A)$. Hence $\rho(A)$ is open.

(c) We prove that the map $\rho(A) \ni\mapsto R_z(A) = (z - A)^{-1} \in L(H)$ is analytic, by taking an arbitrary point $z_0 \in \rho(A)$. As $\rho(A)$ is open, let us take $h \in \mathbb{C}$ with $|h|$ small enough such that $z = Z_0 + h \in \rho(A)$. Let $T := z_0 - A$, and note that $||T|| \neq 0$. Then observe that $z - A = T + h$, whence

$$\begin{aligned}R_{z_0+h} &= (z - A)^{-1} \\ &= (h + T)^{-1} \\ &= T^{-1} - h(T^{-1})^2 + h^2(T^{-1})^3 + ... \\ &= \sum_{n=0}^{\infty} (-h)^n (T^{-1})^{n+1}\end{aligned}$$

where the expansion is again justified as long as we take $|h| < ||T^{-1}||$. Now observe that for $n \geq 2$ the sum in this expansion is of order $\mathcal{O}(h)$ as

$$|| \sum_{n=2}^{\infty} (-h)^n (T^{-1})^{n+1} || \leq \sum_{n=2}^{\infty} |h|^n ||T^{-1}||^{n+1} = \frac{|h|^2 ||T^{-1}||^3}{1 - |h|||T^{-1}||}. \tag{34}$$

We conclude that

$$\lim_{|h|\downarrow 0} \frac{||R_z(A) - R_{z_0}(A)||}{|h|} = \lim_{|h|\downarrow 0} \frac{||(h+T)^{-1} - T^{-1}||}{|h|} = ||T^{-2}|| = ||(z_0 - A)^{-2}||. \tag{35}$$

We are done when we show that $z \mapsto R_z$ is also continous (as then $R_z \circ R_z$) is continuous as a composition of continuous functions in the last line above). But this also follows immediately from (34) and exercise **(a)**. We see that for $|h| > 0$ small and $z := z_0 + h$ that

$$\begin{aligned} ||R_z - R_{z_0}|| &\leq |h|||R_{z_0} \circ R_z|| \\ &\leq |h|||T^{-1}|| \left(\sum_{n=0}^{\infty} |h|^n (T^{-1})^{n+1} \right) \\ &= \left(\sum_{n=1}^{\infty} |h|^n (T^{-1})^{n+1} \right). \end{aligned}$$

This series is absolutely convergent and of order $\mathcal{O}(h)$ again, thus

$$\lim_{|h|\downarrow 0} ||R_{z_0+h} - R_{z_0}|| = 0, \tag{36}$$

and $z \mapsto R_z$ is continuous, and hence so is $R_z \circ R_z$. We conclude that the r.h.s in (35) is continuous, and therefore $z \mapsto R_z$ complex differentiable and analytic.

4.4. Heat equation and the exponential map.

(a) The "neat" framework for this exercise would be *unbounded* Borel functional calculus, however theorem theorem $(T.11)$ gives us a "budget" version of doing this. As *unbounded* Borel functional calculus has only briefly been mentioned in the lectures, we will opt to solve the exercise with theorem $(T.11)$ Furthermore we shift notation to from u to v in the solution as we denote the unitary operator we are going to use with U.
Let $U : H \to L^2(M, d\mu)$ be the unitary transformation from H to a finite measure space (M, μ) such that for all $v \in D(A) \subset H$ we have

$$A(v) = U^{-1} \circ T_g \circ U(v), \tag{37}$$

where $g : M \to \mathbb{R}$ is a measurable function and $T_g : L^2(M, d\mu) \supset D(T_g) \to L^2(M, d\mu)$ where we recall that $D(T_g) = U(D(A))$.
Let $v \in D(A)$ and let us first observe the following:

$$||v||_H^2 = ||Uv||^2_{L^2(M,d\mu)}, \tag{38}$$

as U is unitary. We conclude therfore that

$$\int_M |(Uv)(x)|^2 d\mu(x) < \infty \tag{39}$$

where we emphasize that $Uv \in L^2(M, d\mu)$ for all $v \in H$. Likewise, if $v \in D(A)$, we have

$$\begin{aligned} ||Av||_H^2 &= ||U^{-1}T_g Uv||_H^2 \\ &= ||T_g Uv||^2_{L^2(M,d\mu)}. \end{aligned}$$

Therefore as $v \in D(A) \implies ||Av||_H < \infty$, and we see that

$$\int_M |g(x)|^2 |Uv(x)|^2 d\mu(x) < \infty. \tag{40}$$

We have seen in class as well that

$$e^{-tA} = U^{-1} \circ e^{-tT_g} \circ U, \tag{41}$$

we also know that

$$\operatorname{ess\,ran} g = \sigma(A). \tag{42}$$

Therefore if $\sigma(A) \subseteq [C, \infty)$ we know that

$$g(x) > C \ \ \mu - \text{a.e.}. \tag{43}$$

Therefore, for all $v_0 \in D(A)$ we have that

$$\begin{aligned} ||v_t||_H^2 &= ||e^{-tA}v_0||_H^2 \\ &= ||U^{-1}e^{-tg}U(v_0)||_H^2 \\ &= ||e^{-tg}U(v_0)||^2_{L^2(M,d\mu)} \\ &= \int_M e^{-2tg(x)}|Uv_0(x)|^2 d\mu(x) \\ &\leq e^{-2tC}\int_M |Uv_0(x)|^2 d\mu(x) \\ &= e^{-2tC}||Uv_0||^2_{L^2(M,d\mu)} \\ &= e^{-2tC}||v_0||_H^2, \end{aligned}$$

where we used that $y \mapsto e^{-ty}$ is monotously decreasing in y for $t \geq 0$. Taking the square root in the above we have $||v_t||_H \leq e^{-tC}||v_0||_H$ for all $t \geq 0$ as required. Notice that this implies that if $v_0 \in D(A)$ then $v_0 \in D(e^{-tA})$ for all $t \geq 0$. Let us also now here that if $v_0 \in D(A)$ then $v_t \in D(A)$ for all $t \geq 0$. From the unitary equivalence relation $U^{-1} \circ A \circ U = T_g$ and the fact that $D(T_g) = U(D(A))$ we know that given $v_0 \in D(A)$ (so $||gUv_0||_{L^2(M,d\mu)} < \infty$), we will have that $v_t \in D(A)$ if and only if $gUv_t \in T_g$, i.e. iff $||T_gUv_t||_{L^2(M,d\mu)}$. As $v_t = e^{-tA}v_0 = U^{-1}e^{-tg(x)}Uv_0$ we can further reduce this to the statement that $v_t \in D(A)$ iff $||gUU^{-1}e^{-tg(x)}Uv_0||_{L^2(M,d\mu)} < \infty$ under the assumption that $v_0 \in D(A)$. But this follows immediately:

$$\begin{aligned}
||ge^{-tg}U(v_0)||^2_{L^2(M,d\mu)} &= \int_M e^{-2tg(x)}|g(x)|^2|Uv_0(x)|^2 d\mu(x) \\
&\leq e^{-2tC}\int_M |g(x)|^2|Uv_0(x)|^2 d\mu(x) \\
&= e^{-2tC}||T_gUv_0||^2_{L^2(M,d\mu)} < \infty.
\end{aligned}$$

Thus we see that $||Av_t||^2_H = ||U^{-1}T_g e^{-tg}U(v_0)||^2_H < e^{-2tC}||Av_0||^2_H < \infty$ for all $t \geq 0$ provided that $v_0 \in D(A)$.

(b) We will prove in this exercise only that $v_t \in C^0([0,\infty), D(A))$ with $D(A)$ equipped with the norm $||u||_{D(A)} = ||u||_H + ||Au||_H$ and differ differentiability to the last exercise. Recall that $D(A)$ with this norm is closed as A is self adjoint and thus Hilbert. We will only prove continuity in $t = 0$, the proof for other t is analogous due to the semi-group property of the family $\{e^{-tA}\}_{t\geq 0}$, i.e. $e^{-(t+s)A} = e^{-tA} \circ e^{-sA}$.
Thus, let $v_0 \in D(A)$; we want to show

$$\lim_{t\searrow 0} ||v_t - v_0||_{D(A)} = \lim_{t\searrow 0} (||v_t - v_0||_H + ||Av_t - Av_0||_H) = 0. \tag{44}$$

Let us consider $||v_t - v_0||_H$ first. We know, by the same steps as above that

$$\begin{aligned}
\lim_{t\searrow 0} ||v_t - v_0||^2_H &= \lim_{t\searrow 0} ||(e^{-tA} - 1)v_0||^2_H \\
&= \lim_{t\searrow 0} ||(e^{-tg(x)} - 1)Uv_0||^2_{L^2(M,d\mu)} \\
&= \lim_{t\searrow 0} \int_M (e^{-tg(x)} - 1)^2 |Uv_0(x)|^2 d\mu(x) \\
&= \lim_{t\searrow 0} \int_M (e^{-2tg(x)} - 2e^{-tg(x)} + 1)|Uv_0(x)|^2 d\mu(x)
\end{aligned}$$

As we proved already that $Uv_0 \in L^2(M, d\mu)$ we see that the above integrand is dominated by $(e^{-t2C} + 2e^{-tC} + 1)||Uv_0||_{L^2(M,d\mu)}$, where we flip the sign on the cross term. Moreover we have that $\lim_{t\searrow 0}(e^{-tg(x)}-1)^2|Uv_0(x)|=0$ pointwise μ-a.e.. With dominated convergence we conclude that

$$\lim_{t\searrow 0}||v_t-v_0||_H=0. \tag{45}$$

Let us now consider $||Av_t-Av_0||_H$ for $v_0\in D(A)$. The calculation is analogous, we have

$$\begin{aligned}\lim_{t\searrow 0}||Av_t-Av_0||_H^2&=\lim_{t\searrow 0}||T_g e^{-tg(x)}Uv_0-T_gUv_0||^2_{L^2(M,d\mu)}\\&=\lim_{t\searrow 0}\int_M\left|e^{-tg(x)}-1\right|^2|g(x)|^2|Uv_0(x)|^2d\mu(x).\end{aligned}$$

As $v_0\in D(A)$ we know that $gUv_0\in L^2(M,d\mu)$. Therefore the above integral is dominated by $(e^{-2tC}+2e^{-tC}+1)||T_gUv_0||_{L^2(M,d\mu)}=(e^{-2tC}+2e^{-tC}+1)||Av_0||_H$. Pointwise we again have $\lim_{t\searrow 0}|e^{-tg(x)}-1||Uv(x)|=0$ and with dominated convergence we can conclude that

$$\lim_{t\searrow 0}||Av_t-Av_0||_H=0. \tag{46}$$

(c) We will now prove that $v_t\in C^1([0,\infty),H)$ and moreover that $\frac{d}{dt}v_t=-Av_t$. We will again prove this only for $t=0$ the rest is anologous by the semi-group property. Specifically, we want to prove that for $v_0\in D(A)$

$$\lim_{t\searrow 0}\left|\left|\frac{v_t-v_0}{t}--Av_0\right|\right|_H=\lim_{t\searrow 0}\left|\left|\frac{e^{-tA}v_0-v_0}{t}+Av_0\right|\right|_H=0. \tag{47}$$

Now, we see that

$$\begin{aligned}\lim_{t\searrow 0}\left|\left|\frac{e^{-tA}v_0-v_0}{t}+Av_0\right|\right|_H&=\lim_{t\searrow 0}\left|\left|\frac{(e^{-tg(x)}-\mathbf{1})}{t}Uv_0+T_gUv_0\right|\right|_{L^2(M,d\mu)}\\
&=\left(\int_M\left|\frac{e^{-tg(x)}-1}{t}+g(x)\right|^2|Uv_0(x)|^2\,d\mu(x)\right)^{1/2}\\
&=\left(\int_M\left(\left|\frac{e^{-tg(x)}-1}{t}\right|+|g(x)|\right)^2|Uv_0(x)|^2\,d\mu(x)\right)^{1/2}\\
&=\left(\int_M\left(\left|\frac{e^{-tg(x)}-1}{t}\right|^2+2|g(x)|\left|\frac{e^{-tg(x)}-1}{t}\right|+|g(x)|^2\right)|Uv_0(x)|^2\,d\mu(x)\right)^{1/2}\\
&=\left(\int_M 2\left(\left|\frac{e^{-tg(x)}-1}{t}\right|^2+|g(x)|^2\right)|Uv_0(x)|^2\,d\mu(x)\right)^{1/2},\end{aligned}$$

where we use Young's inequality to absorb the cross term into the squared terms. As we know that $||g(x)Uv_0(x)||_{L^2(M,d\mu)}=||Av_t||_H\le e^{-2tC}||Av_0||_H<\infty$, we are done if we can we can dominate the first term uniformly around $t = 0$. Therefore let $\delta > 0$, and fix $x \in M$. By the intermediate value theorem, we know that for all $0 \leq t < \delta$ there exists a $0 \leq \xi_0 < t$ such that

$$e^{-tg(x)} - 1 = g(x)(e^{-\xi_0 g(x)})(t - 0). \tag{48}$$

Thus we see that

$$\left|\frac{e^{-tg(x)} - 1}{t}\right|^2 = \left|g(x)(e^{-\xi_0 g(x)})\right|^2 < e^{-2\xi_0 C}|g(x)|^2 < e^{-2\delta C}|g(x)|^2 \tag{49}$$

Thus we see that we can dominate the integrand for $t \in [0, \delta)$ uniformly as

$$2\left(\left|\frac{e^{-tg(x)} - 1}{t}\right|^2 + |g(x)|^2\right)|Uv_0(x)|^2 \leq 2(e^{-2\delta C} + 1)|g(x)|^2|Uv_0(x)|^2, \tag{50}$$

which is integrable as $|g(x)|^2|Uv_0(x)|^2$ is. As

$$\lim_{t\searrow 0}\left|\frac{e^{-tg(x)} - 1}{t} + g(x)\right|^2 |Uv_0(x)|^2 = |-g(x) + g(x)|^2\, |Uv_0(x)|^2 = 0$$

pointwise μ-almost everywhere, we conclude again with dominated convergence that

$$\lim_{t\searrow 0}\left|\left|\frac{e^{-tA}v_0 - v_0}{t} + Av_0\right|\right|_H = 0. \tag{51}$$

For the other points $t \neq 0$ we can similarly deduce that $\frac{d}{dt}v_t = Av_t$. We finally note that Av_t is continuous for $t \geq 0$ as we have proven this already in part **(b)** (we know that $\lim_{t\searrow 0}||Av_t - Av_0||_H = 0$, for $t = 0$, and for $t \neq 0$ the proof is analogous. We conclude that $v_t \in C^1([0, \infty), H)$.