5.1. The p-energy functional.

Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be open, bounded and regular, $2 \leq p < \infty$ and $g \in C^2(\partial\Omega)$. Consider

$$E_p(u) := \int_\Omega |\nabla u|^p dx, \quad \text{and} \quad \mathfrak{U} := \{u \in C^2(\bar{\Omega}) \mid u|_{\partial\Omega} = g. \tag{1}$$

(a) Suppose $u_1, u_2 \in \mathfrak{U}$ both satisfy

$$E_p(u_1) = E_p(u_2) = m =: \inf_{v \in \mathfrak{U}} E_p(v). \tag{2}$$

Since for $p \geq 2$ the mapping $\mathbb{R}^n \ni v \mapsto |v|^p$ is strictly convex, that is, we have

$$\left|\frac{v_1 + v_2}{2}\right|^p < \frac{|v_1|^p + |v_2|^p}{2}, \tag{3}$$

for every $v_1, v_2 \in \mathbb{R}^n$ with $v_1 \neq v_2$. If $\nabla u_1 \neq \nabla u_2$ in a set of postiive measure, then we ahve

$$E_p\left(\frac{u_1 + u_2}{2}\right) = \int_\Omega \left|\frac{\nabla u_1 + \nabla u_2}{2}\right|^p dx \leq \int_\Omega \frac{|\nabla u_1|^p + |\nabla u_2|^p}{2} dx = m, \tag{4}$$

which is a contradiction to u_1 and u_2 being minimizers of E_p.
Consequently, $\nabla u_1 = \nabla u_2$ a.e. on Ω. Then by continuity $\nabla u_1 = \nabla u_2$ on Ω, which means that $u_1 - u_2$ is constant on every connected component of Ω. Since $(u_1 - u_2)|_{\partial\Omega} = 0$ the conclusion now follows from the lemma below with $u = u_1 - u_2$.

Lemma. *If $u \in C^2(\bar{\Omega})$ with $u|_{\partial\Omega} = 0$ satisfies $\Delta u = 0$ in Ω, then $u \equiv 0$.*

Proof. First note if $u \in C^2(\bar{\Omega})$ with $u|_{\partial\Omega} = 0$ we can apply integration by parts and we get

$$\int_\Omega |\nabla u|^2 dx = -\int_\Omega u \Delta u dx \leq \int_\Omega |u||\Delta u| dx \leq \left(\int_\Omega u^2 dx\right)^{1/2} \left(\int_\Omega |\Delta u|^2 dx\right)^{1/2}, \tag{5}$$

where we used Cauchy-Schwarz in the final inequality. Now we note that if $u \in C^2(\bar{\Omega})$ with $u|_{\partial\Omega} = 0$ satisfies $\Delta u = 0$ on Ω, then

$$\int_\Omega |\nabla u|^2 dx = -\int_\Omega u \underbrace{\Delta u}_{=0} \, dx = 0. \tag{6}$$

Since $|\nabla u(x)|^2 \geq 0$ for every $x \in \Omega$ we know that $|\nabla u|^2 \equiv 0$ on Ω. Since Ω is connected this means that u is constant on Ω. From continuity it follows that $u(x) = u|_{\partial\Omega} = 0$ as required. $\square$

(b) Suppose $u \in \mathfrak{U}$ is a minimizer of E_p. Let $\phi \in C^2(\bar{\Omega})$ satisfy $\phi|_{\partial\Omega} = 0$. Then $u + t\phi \in \mathfrak{U}$ for every $t \in \mathbb{R}$. Moreover,

$$\frac{d}{dt}\int_\Omega |\nabla u + t\nabla\phi|^p \, dx = p\int_\Omega |\nabla u + t\nabla\phi|^{p-2}(\nabla u + t\nabla\phi)\cdot\nabla\phi \, dx. \tag{7}$$

In particular,

$$0 = \frac{d}{dt}\bigg|_{t=0}\int_\Omega |\nabla u + t\nabla\phi|^p \, dx = p\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi \, dx = -p\int_\Omega \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)\nabla\phi \, dx$$

for every $C^2(\bar{\Omega})$ with $\phi|_{\partial\Omega} = 0$. Hence, by the fundamental lemma of the calculus of variations, $-\operatorname{div}(|\nabla u|^{p-2}|\nabla u|) = 0$ a.e. in Ω. By ocntinuity, $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ on Ω.

(c) For every $u \in C^2(\bar{\Omega})$ with $u|_{\partial\Omega} = 0$ we have that

$$\begin{aligned}\int_\Omega |\nabla u|^p \, dx = \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla u \, dx &= -\int_\Omega \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) u \, dx \\ &= -\int_\Omega \left((p-2)|\nabla u|^{p-4}\left(D^2u(\nabla u,\nabla u)\right) + |\nabla u|^{p-2}\Delta u\right) u \, dx\end{aligned}$$

$$\leq (p-2+\sqrt{n})\int_\Omega |\nabla u|^{p-2}|D^2u||u| \, dx,$$

where $(\Delta u)^2 \leq n|D^2u|^2$ is used. Indeed, with $\frac{\partial u}{\partial x_j} =: u_j$ and $\frac{\partial^2 u}{\partial x_j \partial x_k} =: u_{jk}$ we have

$$\begin{aligned}\left|D^2u(\nabla u,\nabla u)\right| &= \left|\sum_{j=1}^n u_j \sum_{k=1}^n u_{jk}u_k\right| \leq \left(\sum_{j=1}^n u_j^2\right)^{1/2}\left(\sum_{j=1}^n\left(\sum_{k=1}^n u_{jk}u_k\right)^2\right)^{1/2} \\ &\leq \left(\sum_{j=1}^n\left(\sum_{k=1}^n u_{jk}^2\right)^2\left(\sum_{k=1}^n u_k^2\right)^2\right)^{1/2} = |\nabla u|^2\left(\sum_{j=1}^n\sum_{k=1}^n u_{jk}^2\right)^{1/2} = |\nabla u|^2|D^2u|,\end{aligned}$$

so we have

$$\left(\frac{\nabla u}{n}\right)^2 = \left(\frac{u_{11}+...+u_{nn}}{n}\right)^2 \leq \frac{u_{11}^2+...+u_{nn}^2}{n} \leq \frac{1}{n}\sum_{j=1}^n\sum_{k=1}^n u_{jk}^2 = \frac{1}{n}|D^2u|^2.$$

Using Hölder's inequality with $1 = \frac{p-2}{p} + \frac{1}{p} + \frac{1}{p}$, we obtain

$$\int_\Omega |\nabla u|^p dx \leq (p-2+\sqrt{n})\left(\int_\Omega |\nabla u|^p \, dx\right)^{\frac{p-2}{p}}\left(\int_\Omega |D^2u|^p \, dx\right)^{\frac{1}{p}}\left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}}.$$

From this it follows that

$$(int_\Omega|\nabla u|^p dx)^{\frac{2}{p}} \leq (p-2+\sqrt{n})\left(\int_\Omega |D^2u|^p \, dx\right)^{\frac{1}{p}}\left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}}, \tag{8}$$

whence

$$\int_\Omega |\nabla u|^p dx \le (p-2+\sqrt{n})^{\frac{p}{2}} \left(\int_\Omega |D^2u|^p\,dx\right)^{\frac{1}{2}} \left(\int_\Omega |u|^p\,dx\right)^{\frac{1}{2}}. \tag{9}$$

5.2. Weak derivative in $L^p(\Omega)$

(a) Let $u \in L^1_{\mathrm{loc}}(\Omega)$. Given $1 < p \le \infty$, let $1 \le q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $D^\alpha u$ exists as weak derivative in $L^p(\Omega)$. Let $\varphi \in C_c^\infty(\Omega)$ be arbitrary. Then,

$$\left|\int_\Omega u D^\alpha \varphi\,dx\right| = \left|(-1)^{|\alpha|}\int_\Omega (D^\alpha u)\varphi\,dx\right| \le \|D^\alpha u\|_{L^p(\Omega)}\|\varphi\|_{L^q(\Omega)}$$

by Hölder's inequality which proves the first claim with constant $C = \|D^\alpha u\|_{L^p(\Omega)}$. Conversely, suppose

$$\forall \varphi \in C_c^\infty(\Omega): \quad \left|\int_\Omega u\,D^\alpha\varphi\,dx\right| \le C\|\varphi\|_{L^q(\Omega)}.$$

Then, since $C_c^\infty(\Omega)$ is dense in $L^q(\Omega)$ for $q < \infty$, the map

$$f\colon \varphi \mapsto (-1)^{|\alpha|}\int_\Omega u\,D^\alpha\varphi\,dx$$

defines a continuous linear functional $f \in (L^q(\Omega))^*$. Since $(L^q(\Omega))^*$ for $1 \le q < \infty$ is isometrically isomorphic to $L^p(\Omega)$, there exists $g \in L^p(\Omega)$ such that

$$\forall \varphi \in L^q(\Omega): \quad f(\varphi) = \int_\Omega g\varphi\,dx.$$

By definition of f it follows that $g \in L^p(\Omega)$ is the weak derivative $D^\alpha u$ of u.

(b) Let $u = \chi_{]0,1[}$ and $\varphi \in C_c^\infty(\mathbb{R})$. Then

$$\left|\int_\mathbb{R} u\,\varphi'\,dx\right| = \left|\int_0^1 \varphi'\,dx\right| = \left|\varphi(1) - \varphi(0)\right| \le 2\|\varphi\|_{L^\infty(\mathbb{R})}.$$

The function u restricted to $\mathbb{R} \setminus \{0,1\}$ is differentiable with vanishing derivative. In particular, if u had a weak derivative $u' \in L^1_{\mathrm{loc}}(\mathbb{R})$, then $u' = 0$ almost everywhere. A contradiction arises for test functions $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi(0) \ne \varphi(1)$ via

$$0 = \int_\mathbb{R} u'\varphi\,dx = -\int_\mathbb{R} u\,\varphi'\,dx = -\int_0^1 \varphi'\,dx = \varphi(0) - \varphi(1).$$

5.3. The ice-cream cone

(a) Fix any $\varphi \in C_c^\infty(\Omega)$ and pick a small positive constant $0 < \varepsilon < 1$. Then, define

$$I_\varepsilon := -\int_{\Omega \setminus B_\varepsilon(0)} u(x,y) \frac{\partial \varphi}{\partial x}(x,y)\, dx\, dy = -\int_{\Omega \setminus B_\varepsilon(0)} \left(1 - \sqrt{x^2+y^2}\right) \frac{\partial \varphi}{\partial x}(x,y)\, dx\, dy$$

and

$$J_\varepsilon := -\int_{B_\varepsilon(0)} u(x,y) \frac{\partial \varphi}{\partial x}(x,y)\, dx\, dy = -\int_{B_\varepsilon(0)} \left(1 - \sqrt{x^2+y^2}\right) \frac{\partial \varphi}{\partial x}(x,y)\, dx\, dy.$$

Clearly,

$$|J_\varepsilon| = \left| \int_{B_\varepsilon(0)} \left(1 - \sqrt{x^2+y^2}\right) \frac{\partial \varphi}{\partial x}(x,y)\, dx\, dy \right| \leq \pi \|\nabla \varphi\|_{L^\infty(\Omega)} \varepsilon^2 \to 0$$

as $\varepsilon \to 0^+$. On the other hand, since u is smooth on $\Omega \setminus B_\varepsilon(0)$, we can integrate by parts in the integral I_ε to get

$$\begin{aligned} I_\varepsilon &= \int_{\partial B_\varepsilon(0)} \left(1 - \sqrt{x^2+y^2}\right) \varphi(x,y) \frac{x}{\varepsilon}\, d\sigma - \int_{\Omega \setminus B_\varepsilon(0)} \frac{x}{\sqrt{x^2+y^2}} \varphi(x,y)\, dx\, dy \\ &= (1-\varepsilon) \int_{\partial B_\varepsilon(0)} \varphi(x,y) \frac{x}{\varepsilon}\, d\sigma - \int_{\Omega \setminus B_\varepsilon(0)} \frac{x}{\sqrt{x^2+y^2}} \varphi(x,y)\, dx\, dy. \end{aligned}$$

Notice that

$$\left| (1-\varepsilon) \int_{\partial B_\varepsilon(0)} \varphi(x,y) \frac{x}{\varepsilon}\, d\sigma \right| \leq 2\pi \|\varphi\|_{L^\infty(\Omega)} (1-\varepsilon)\varepsilon \to 0$$

as $\varepsilon \to 0^+$. Moreover, since

$$\begin{aligned} \left| \int_\Omega \frac{x}{\sqrt{x^2+y^2}} \varphi(x,y)\, dx\, dy \right| &\leq \|\varphi\|_{L^\infty(\Omega)} \left(\int_0^{2\pi} |\cos\theta|\, d\theta \right) \cdot \left(\int_0^1 r dr \right) \\ &= 2\|\varphi\|_{L^\infty(\Omega)} < +\infty, \end{aligned}$$

by dominated convergence we get

$$-\int_{\Omega \setminus B_\varepsilon(0)} \frac{x}{\sqrt{x^2+y^2}} \varphi(x,y)\, dx\, dy \to -\int_\Omega \frac{x}{\sqrt{x^2+y^2}} \varphi(x,y)\, dx\, dy,$$

as $\varepsilon \to 0^+$. Thus,

$$I_\varepsilon + J_\varepsilon \to -\int_\Omega \frac{x}{\sqrt{x^2+y^2}} \varphi(x,y)\, dx\, dy.$$

But since

$$I_\varepsilon + J_\varepsilon = -\int_\Omega u(x,y)\frac{\partial\varphi}{\partial x}(x,y)\,dx\,dy,$$

for every $0<\varepsilon<1$, by uniqueness of the limit we obtain

$$-\int_\Omega u(x,y)\frac{\partial\varphi}{\partial x}(x,y)\,dx\,dy = -\int_\Omega \frac{x}{\sqrt{x^2+y^2}}\varphi(x,y)\,dx\,dy.$$

Since Ω has finite measure, it holds that $L^\infty(\Omega)\hookrightarrow L^p(\Omega)$ continuously for every $p\in[1,\infty)$, and so it follows that such weak partial derivative of u exists in $L^p(\Omega)$ for every $p\in[1,\infty]$ and is given by

$$\frac{\partial u}{\partial x}(x,y) = -\frac{x}{\sqrt{x^2+y^2}} \qquad \text{a.e. on } \Omega.$$

Analogous conclusions hold for the weak partial derivative with respect to y of u on Ω, which is given by

$$\frac{\partial u}{\partial y}(x,y) = -\frac{y}{\sqrt{x^2+y^2}} \qquad \text{a.e. on } \Omega.$$

(b) First, notice that

$$|\nabla u|^2 = \left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 = 1 \qquad \text{a.e. on } \Omega.$$

Thus,

$$\|\nabla u\|_{L^p(\Omega)} = \pi^{1/p} \qquad \forall\, p\in[1,\infty)$$

and

$$\|\nabla u\|_{L^\infty(\Omega)} = 1.$$

5.4. A closedness property.

(a) Given $I := \,]a,b[$ for $-\infty\le a<b\le\infty$ and $1<p\le\infty$, let $u\in L^p(I)$ and let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $W^{1,p}(I)$ satisfying $\|u_k-u\|_{L^p(I)}\to 0$ as $k\to\infty$. Let u_k' be the weak first derivative of u_k. By assumption, the sequence $(u_k')_{k\in\mathbb{N}}$ is bounded in $L^p(I)$.

Case $1<p<\infty$. In this case, the space $L^p(I)$ is reflexive and the Eberlein–Šmulyan Theorem applies: $(u_k')_{k\in\mathbb{N}}$ has a subsequence which converges weakly in $L^p(I)$. Let

$g \in L^p(I)$ be the corresponding weak limit and $\Lambda \subset \mathbb{N}$ the subsequence's indices. Since for any $\varphi \in C_c^\infty(I)$, the maps $L^p(I) \to \mathbb{R}$ given by $f \mapsto \int_I f\varphi\,dx$ or by $f \mapsto -\int_I f\varphi'\,dx$ are elements of $(L^p(I))^*$ and since $\|u_k - u\|_{L^p} \to 0$ implies $u_k \xrightarrow{\mathrm{w}} u$, we have by definition of weak convergence

$$-\int_I u\varphi'\,dx = \lim_{\Lambda\ni k\to\infty}\Big(-\int_I u_k\varphi'\,dx\Big) = \lim_{\Lambda\ni k\to\infty}\Big(\int_I u_k'\varphi\,dx\Big) = \int_I g\varphi\,dx$$

for any $\varphi \in C_c^\infty(I)$. Hence, $g \in L^p(I)$ is indeed the weak derivative of $u \in L^p(I)$ and $u \in W^{1,p}(I)$ follows.

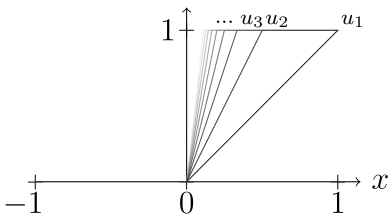
Case $p = \infty$. Since $L^1(I)$ is separable, the Banach–Alaoglu Theorem applies: $(u_k')_{k\in\mathbb{N}}$ being bounded in $L^\infty(I) \cong (L^1(I))^*$ has a subsequence (given by $\Lambda \subset \mathbb{N}$) which weak*-converges to some $g \in (L^1(I))^*$. For any $\varphi \in C_c^\infty(]0,1[) \subset L^1(]0,1[)$,

$$-\int_I u\varphi'\,dx = \lim_{\Lambda\ni k\to\infty}\Big(-\int_I u_k\varphi'\,dx\Big) = \lim_{\Lambda\ni k\to\infty}\Big(\int_I u_k'\varphi\,dx\Big) = \int_I g\varphi\,dx$$

follows as in part (a) with the only difference, that the last identity comes from weak*-convergence rather than weak convergence. Hence, $g \in (L^1(I))^* \cong L^\infty(I)$ is indeed the weak derivative of $u \in L^\infty(I)$ and $u \in W^{1,\infty}(I)$ follows.

(b) The assumption $p \neq 1$ in part (a) is necessary. Consider $I =]-1,1[$ and $u = \chi_{]0,1[} \in L^1(I)$. For every $k \in \mathbb{N}$ let $u_k\colon I \to \mathbb{R}$ be given by

$$u_k(x) = \begin{cases} 0, & \text{for } -1 < x \le 0,\\ kx, & \text{for } 0 < x \le \frac{1}{k},\\ 1, & \text{for } \frac{1}{k} < x \le 1.\end{cases}$$



Then, $u_k \in W^{1,1}(I)$ with $\|u_k\|_{L^1} = 1 - \frac{1}{2k}$ and $\|u_k'\|_{L^1} = \frac{1}{k}k = 1$. Moreover, there holds $\|u_k - u\|_{L^1} = \frac{1}{2k} \to 0$ as $k \to \infty$. However, $u \notin W^{1,1}(I)$, otherwise u would have a continuous representative.

Remark. This is not a counterexample in the case $p > 1$, where $\|u_k'\|_{L^p} = (\frac{1}{k}k^p)^{\frac{1}{p}} \to \infty$.