6.1. The Dirichlet problem on an interval.

(a) Existence of a solution $u \in H_0^1(I)$ can be quickly deduced using Riesz' representation theorem. More specifically, we have the Poincare inequality for $\Omega \subset]0, L[\times\mathbb{R}^{n-1}$ that tells us that

$$\int_\Omega |u|^2 dx \le L^2 \int_\Omega |\nabla u|^2 dx, \tag{1}$$

for all $u \in C_c^\infty(\Omega)$. In particular this means for $I = (a, b)$ that there exists a constant $C > 0$ such that

$$||u||_{L^2(I)} \le C||\nabla u||_{L^2(I)}. \tag{2}$$

This shows that for $u \in C_c^\infty(I)$ the norm $|| \cdot ||_\nabla$ defined through

$$||u||_\nabla = \int_I |\nabla u|^2 dx \tag{3}$$

is equivalent[1] to the standard H^1 norm given by $||u||_{H^1(I)} = ||u||_{L^2} + ||u||_\nabla$ for all $u \in C_c^\infty(I)$. Therefore, the *closure* of $C_c^\infty(I)$ with respect to $|| \cdot ||_\nabla$ will again yield $H_0^1(I)$, which will again be a Hilbert space with respect to the scalar product

$$\langle u, v\rangle_\nabla = \int_I u'v'\, dx. \tag{4}$$

We then close out the argument by noting that the map $L_f : C_c^\infty(I) \to \mathbb{C}$ given by

$$L_f(\phi) = \int_I f\phi\, dx. \tag{5}$$

is bounded with respect to the $|| \cdot ||_\nabla$ norm, for any $f \in C^0(\bar{I})$ as we have

$$|L_f(\phi)| \le ||f||_{L^2(I)}||\phi||_{L^2(\bar{I})} \le C||f||_{L^2(I)}||\nabla\phi||_{L^2(I)}, \tag{6}$$

where we use the Cauchy-Schwarz inequality *and* again the Poincaré inequality. Thus L_f is bounded hence continuous, and can be extended from $C_c^\infty(I)$ to $H_0^1(I)$ by density. Then using that $H_0^1(I)$ is a Hilbert space, Riesz' represenation theorem gives us the existence of a $u \in H_0^1(I)$ such that

$$L_f(v) = \langle u, v\rangle_\nabla \text{ for all } v \in H_0^1(I), \tag{7}$$

which yields us exactly that

$$\int_I fv\, dx = \int_I u'v' dx \tag{8}$$

for all $v \in H_0^1(I)$.

[1]In fact this equivalence is so well-known that almost always the $||u||_\nabla$ norm is referred to as the standard H^1 norm on the interval $I = (a, b)$. We just denote it with different notation here to emphasize that there are indeed two equivalent H^1 norms that one could define on the interval.

(b) We will first show that $u \in C^2(\bar{I})$. As (8) holds for $v \in C_c^\infty(I)$ in particular we know that $u' \in L^2(I)$ has the weak solution

$$(u')' = -f \in C^0(\bar{I}) \hookrightarrow L^\infty(I), \tag{9}$$

i.e. $u \in W^{1,\infty}$. Using a combination of theorem $T.18$ na dthe fundamental theorem, we see that

$$u(x) = u'(x_0) - \int_{x_0}^{x} f(t)dt \in C^1(\bar{I}). \tag{10}$$

Thus $u \in C^2(\bar{I})$ and hence u solves

$$-u''(x) = f(x) \tag{11}$$

in the classical sense. To check boundary conditions, we use that $H_0^1(I) = \overline{C_c^\infty(I)}^{||\cdot||_\nabla}$. Hence we choose a $(u_k)_{k\in\mathbb{N}} \subset C_c^\infty(I)$ with

$$||u_k - u||_{L^\infty} \leq C||u_k - u||_{H^1} \to 0 \text{ as } k \to \infty. \tag{12}$$

Thus we have for arbitrary $k \in \mathbb{N}$ that

$$|u(a)| \leq \underbrace{|u_k(a)|}_{=0} + ||u_k - u||_{L^\infty} \leq C||u_k - u||_{H^1} \to 0 \text{ as } k \to \infty, \tag{13}$$

where the $u_k(a)$ are obviously 0 as the u_k are compactly supported in (a, b). We connclude $u(a) = 0$. Analogously one can show $u(b) = 0$

6.2. The Neumann problem on an interval.

(a) For $x, y \in I$ we have

$$|u(x) - u(y)| \leq \int_I |u'(t)|dt = ||u'||_{L^1}. \tag{14}$$

Therefore it follows that

$$\begin{aligned} |u(x)| = |u(x) - \bar{u}| &= \left| \frac{1}{|I|} \int_I (u(x) - u(y))\, dy \right| \\ &\leq \frac{1}{|I|} \int_I |u(x) - u(y)|\, dy \frac{1}{|I|} ||u'||_{L^1}. \end{aligned}$$

Thus we see that

$$||u||_{L^2}^2 = \frac{1}{|I|^2} ||u'||_{L_1}^2 \int_I 1\, dx,$$

so

$$||u||_{L^2} = \frac{1}{\sqrt{|I|}}||u'||_{L^1}.$$

Now using Hölder we see that

$$||u||_{L^1(I)} = \int_I |u'| \cdot 1dx \leq ||u'||_{L^2}||1||_{L^2} = \sqrt{|I|}||u'||_{L^2}. \tag{15}$$

We conclude that here we have

$$||u||_{L^2} \leq ||u'||_{L^2} \tag{16}$$

as required. Note that this gives an equivalent norm on X (compare this with our considerations regarding the Poincare inequality in the previous exercises. Obviously this norm is induced by

$$(u, v)_X = \int_I u'v'\, dx.$$

As X is a closed subspace of a Hilbert space it is complete, and therefore Hilbert space.

(b) This solution is analogous to the previous exercise. In summary, we show that the functional $L_f : H^1(I) \to \mathbb{C}$ given by

$$L_f(v) = \int_I fv\, dx$$

is bounded in L^2 norm on X, and because of exercise **(a)**, also in H^1 norm on X: we again have

$$|L_f(v)| \leq ||f||_{L^2}||v||_{L^2} \leq ||f||_{L^2}||v'||_{L^2},\ \forall v \in X$$

Thus there exists an $u \in X$ such that

$$(u, v)_X = \int_I u'v'\, dx = L_f(v) = \int_I fv\, dx. \tag{17}$$

as required.

(c) The fact that $u \in C^2(I)$ (regularity) follows also immediately from the previous question as well as the fact that $-u'' = f$ in the strong sense. For the boundary conditions notice then that this combined witht he fact that u satisfies () implies that

$$\begin{aligned} 0 &= \int_I (u'v' - fv)\, dx \\ &= \int_I (-u'' - f)\, dx + u'(b)v(b) - u'(a)v(a),\ \text{for all } v \in H^1(I). \end{aligned}$$

Therefore, as $v \in H^1(I)$ is arbitrary and thus also $v(a)$ and $v(b)$, we conclude that $u'(b) = 0 = u'(a)$.

6.3. Equivalent characterizations of $W^{1,p}(\Omega)$

(**c**) $\implies$ (**b**) Let $\phi \in C_c^\infty(\Omega)$, and let $\Omega' \Subset \Omega$ be such that $\operatorname{supp}\phi \subset \Omega'$. Let $|h| < d(\Omega', \partial\Omega)$, then we get with Hölder the estimate

$$A := \int_\Omega \underbrace{(u(x+h) - u(x))}_{=\tau_h u - u \in L^p} \phi(x)\, dx \leq ||\tau_h - u||_{L^p(I')} \cdot ||\phi|||_{L^q(I')}.$$

On the other hand we get when we substitute $y = x + h$ that

$$A = \int_I (u(y)\phi(y-h) - u(y)\phi(y))\, dy = -\int_I u(y)(\phi(y) - \phi(y-h))\, dy. \tag{18}$$

After division by $|h|$ and the limit $h \to 0$ and using the assumption $||\tau_h u - u||_{L^p(\Omega')} \leq C|h|$ that

$$\left|\int_I u\nabla\phi\, dx\right| = \lim_{h\searrow 0} \left|\int_I u(x)\frac{\phi(x) - \phi(x-h)}{h}\, dx\right| \leq C \cdot ||\phi||_{L^q(\Omega)},$$

with $C > 0$ independent from ϕ.
(**b**) $\implies$ (**a**) If we assume **(b)**[2] to hold then we can continuously extend the map

$$C_c^\infty(\Omega) \ni \phi \mapsto \int_\Omega u\phi'\, dx, \tag{19}$$

continuously to a functional $l \in (L^q(\Omega))^*$ where we use density once again. But then using Riesz representation for $L^q(\Omega)$ we find that there must exist a $g \in L^q(\Omega)$ such that

$$l(\phi) = -\int_I g\phi\, dx \text{ for all } \phi \in L^q(\Omega).$$

In particular it holds that

$$\int_\Omega u\nabla\phi\, dx = l(\phi) = -\int_\Omega g\phi\, dx, \text{ for all } \phi \in C_c^\infty(\Omega),$$

thus $g \in L^p(\Omega)$ is the weak derivative of u, and we conclude $u \in W^{1,p}(\Omega)$.
(**a**) $\implies$ (**c**) Let $u \in C^1(\Omega)$ and $\Omega' \Subset \Omega$. Then for $|h| < d(\Omega', \partial\Omega)$, $x \in \Omega'$ we have

$$\begin{aligned} |\tau_h u(x) - u(x)| = |u(x+h) - u(x)| &= \left|\int_0^1 h \cdot \nabla u(x+th)\, dt\right| \\ &\leq |h| \int_0^1 |\nabla u(x+th)|\, dt. \end{aligned}$$

To finalize the proof we now separate by cases. For $p < \infty$ we have with Hölder that

$$|\tau_h u(x) - u(x)|^p \leq |h|^p \int_0^1 |\nabla u(x+th)|^p\, dt \tag{20}$$

[2] In the original exercise sheet there was a typo on assumption **(b)**.

which combined with Fubini gives us

$$||\tau_h u - u||^p_{L^p(\Omega')} \leq |h|^p \int_{\Omega'} \int_0^1 |\nabla u(x+th)|^p \, dtdx \leq |h|^p ||\nabla u||^p_{L^p(\Omega)}. \tag{21}$$

We then use that for $u \in W^{1,p}(\Omega)$ there is a sequence $(u_k)_{k\in\mathbb{N}} \subset C^1(\Omega) \cap W^{1,p}(\Omega)$ such that $u_k \to u$ in $W^{1,p}(\Omega)$ for $k \to \infty$. As the last equation (21) holds for all these u_k we conclude that it also holds for u in the limit $k \to \infty$.
For $p = \infty$: For $\Omega \Subset \Omega$, choose Ω'' such that $\Omega' \Subset \Omega'' \Subset \Omega$. As $W^{1\infty}(\Omega) \hookrightarrow W^{1,p}(\Omega'')$ for every $p < \infty$, we obtain with (21) and the Hölder inequality for $h < d(\Omega', \partial\Omega''$ the estimate

$$||\tau_h u - u||_{L^p(\Omega')} \leq |h| \cdot ||\nabla u||_{L^p(\Omega'')} \leq |h| \mathrm{Vol}(\Omega'')^{\frac{1}{p}} ||\nabla||_{L^\infty(\Omega)}.$$

Thus for $p \to \infty$ we have

$$||\tau_h u - u||_{L^\infty(\Omega')} \leq |h| \cdot ||\nabla u||_{L^\infty}.$$

A quick inspection of the last part above allows us to deduce that in case **(b)** and **(c)** the constant is given as $C = ||\nabla u||_{L^p(\Omega)}$.

6.4. Chain rule for Sobolev functions

Let us define $v = G \circ u$. Then as $G(0) = 0$ and $|G'(s)| \leq L$ we have

$$|v(x)| = |G(u(x)) - G(0)| \leq L|u(x)| \in L^p(\Omega), \tag{22}$$

and analogously we obtain for $g = (G' \circ u) \cdot \nabla u$ the estimate

$$|g(x)| = |G'(u(x))| \cdot |\nabla u(x)| \leq L|\nabla u(x)| \in L^p(\Omega). \tag{23}$$

We claim that $g = \nabla v$. To prove this claim, let $\phi \in C_c^\infty(\Omega)$. Let $\Omega' \Subset \Omega$ with $\mathrm{supp}(\phi)$. Interpret $u|_{\Omega'}$ as $u \in W^{1,1}(\Omega')$. We take as in the previous exercise a sequence $(u_k)_{k\in\mathbb{N}} \subset C^1(\Omega') \cap W^{1,1}(\Omega')$ with $u_k \to u$ in $W^{1,1}(\Omega')$ as $k \to \infty$. According to the (usual) chain rule we have

$$\int_\Omega G(u_k) \frac{\partial \phi}{\partial x_i} dx = -\int_\Omega G'(u_k) \frac{\partial u_k}{\partial x_i} \phi dx, 1 \leq i \leq n, k \in \mathbb{N}.$$

Furthermore we have

$$\left| \int_\Omega (G(u_k) - G(u)) \frac{\partial \phi}{\partial x_i} dx \right| \leq C||u_k - u||_{L^1(\Omega)} \to 0(k \to infty), \tag{24}$$

as $|G(u_k) - G(u)| \leq L|u_k - u|$, $|\frac{\partial \phi}{\partial x_i}| \leq C$. Analogously we deduce that

$$\begin{aligned}
&\left| \int_\Omega (G(u_k)\frac{\partial u_k}{\partial x_i} - G(u)\frac{\partial u}{\partial x_i})\phi dx \right| \\
&\leq \left| \int_\Omega G(u_k)(\frac{\partial u_k}{\partial x_i} - \frac{\partial u}{\partial x_i})\phi dx \right| + \left| \int_\Omega (G'(u_k) - G'(u))\frac{\partial u}{\partial x_i}\phi dx \right| \\
&\leq LC||u_k - u||_{W^{1,1}(\Omega')} + C\int_{\Omega'} |G'(u_k) - G'(u)| \left|\frac{\partial u}{\partial x_i}\right| dx \to 0 \ \text{ as } \ k \to \infty.
\end{aligned}$$

where we use the dominated convergence for the last integral. Observe that $|G'(u_k) - G'(u)| \leq 2L$ with $G'(u_k) \to G'(u)$ almost everywhere in Ω' and that $\frac{\partial u}{\partial x_i} \in L^1(\Omega')$. It follows that

$$\begin{aligned}
\int_\Omega v\frac{\partial \phi}{\partial x_i}dx &= \int_\Omega G(u)\frac{\partial \phi}{\partial x_i}dx = \lim_{k\to\infty}\int_\Omega G(u_k)\frac{\partial \phi}{\partial x_i}dx \\
&= -\lim_{k\to\infty}\int_\Omega G'(u_k)\frac{\partial u_k}{\partial x_i}\phi dx = \int_\Omega G(u)\frac{\partial u}{\partial x_i}\phi dx = \int_\Omega g_i\phi \, dx.
\end{aligned}$$

With this the claim is proven and hence the exercise.