7.1. A $W^{1,p}$ function that is not in L^∞

Note that we have

$$\partial_i u(x) = -\frac{1}{\log(1+\frac{1}{|x|})}\frac{x_i}{|x|^3}\frac{1}{1+\frac{1}{|x|}}$$

and moreover that

$$\frac{x_i}{|x|^3}\frac{1}{1+\frac{1}{|x|}} = \frac{x_i}{|x|^2}\frac{1}{|x|+1}$$

The term $\frac{1}{|x|+1}$ does not have a singularity at $|x|=0$ and

$$\left|\frac{x_i}{|x|^2}\right| = \frac{1}{|x|}. \tag{1}$$

Therefore we have

$$|\nabla u| \leq \frac{1}{\log(|x|)}\frac{1}{|x|^2} \tag{2}$$

which is integrable after switching to polar coordinates.

7.2. Absolute value of $u \in W^{1,p}$

We note first of all that $f_\epsilon(x) = (x^2+\epsilon)^{1/2} - \epsilon$ is bounded from above by $|x|$ with $\epsilon > 0$ small and that

$$\lim_{\epsilon \searrow 0} f_\epsilon(x) = |x| \tag{3}$$

pointwise. Moreover we have

$$f'_\epsilon(x) = \frac{x}{(x^2+\epsilon^2)^{1/2}} \tag{4}$$

and that

$$|f'_\epsilon(x)| \leq 1. \tag{5}$$

combined with the pointwise limit

$$\lim_{\epsilon \searrow 0} f'_\epsilon(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0. \end{cases} \tag{6}$$

as the limit exists we have that the sequence $f_\epsilon \circ u$ is Cauchy in the L^p norm, hence converges. We note that the gradient of $f_\epsilon \circ u$ is given by

$$\nabla(f_\epsilon \circ u) = (f'_\epsilon \circ u) \cdot \nabla u. \tag{7}$$

where we note that we note that $|\nabla(f_\epsilon \circ u)| \le 2^p C|\nabla u|^p$ for $\epsilon > 0$ small. Therefore in the pointwise limit we have

$$\lim_{\epsilon \searrow 0} \nabla(f_\epsilon \circ u) = \begin{cases} \nabla u & \text{for } u(x) > 0 \\ -\nabla u & \text{for } u(x) < 0. \end{cases} \tag{8}$$

Let us call this last function $\chi_u \nabla u$. We have thus shown that both $f_\epsilon \circ u \to |u|$ and $f_\epsilon \circ \nabla u \to \chi_u \nabla u$ in L^p. It can also easily be shown by testing against any $\phi \in C_c^\infty$ and using the dominated convergence theorem to take the limit $\epsilon \searrow 0$ to show that $\chi_u \nabla u = \nabla |u|$.

7.3. Trace-zero functions in $W^{1,p}$

We denote with $T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ the trace operator.

(a) If $u \in W_0^{1,p}(\Omega)$ then by definition there exist functions $u_n \in C_c^\infty(\Omega)$ such that

$$u_n \to u \text{ in } W^{1,p}(\Omega). \tag{9}$$

As $Tu_m = 0$ on $\partial\Omega$ and T is bounded, we conclude $Tu = u|_{\partial\Omega} = 0$.

(b) i) Let us now assume $Tu = u|_{\partial\Omega} = 0$. Using a partition of unity and flattining out the boundary $\partial\Omega$ we may assume that

$$\begin{cases} u \in W^{1,p}(\mathbb{R}^n_+), \text{ supp}(u) \text{ is compact in } \overline{\mathbb{R}^n_+} \\ Tu = 0 \text{ on } \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}. \end{cases}$$

ii) Since $Tu = 0$ on $\mathbb{R}^{n-1}$, there exist $u_k \in C^1(\overline{\mathbb{R}^n})$ such that

$$u_k \to u \text{ in } W^{1,p}(\mathbb{R}^n_+)$$

and

$$Tu_k \to u_k|_{\mathbb{R}^{n-1}} \text{ in } L^p(\mathbb{R}^{n-1})$$

Now if $x' \in \mathbb{R}^{n-1}$, $x_n \ge 0$ it follows that

$$|u_k(x', x_n)| \le |u_k(x', 0)| + \int_0^{x_n} |\partial_{x_n} u(x', t)| \, dt.$$

Thus

$$\int_{\mathbb{R}^{n-1}} |u_k(x', x_n)|^p dx' \le \int_{\mathbb{R}^{n-1}} |u_k(x', 0)|^p dx' + C x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^n} |Du|^p dx' dt.$$

Thus letting $k \to \infty$ and rescaling we see that

$$\int_{\mathbb{R}^{n-1}} |u_k(x', x_n)|^p dx' \leq C x_n^{p-1} \int_0^{x_n} \int_{\mathbb{R}^n} |Du|^p dx' dt.$$

for $x_n > 0$ almost everywhere.
iii)-iv) For η as given in the exercise we immediately see that

$$\begin{cases} \partial_{x_n} u_m & = \partial_{x_n} u(1 - \eta(mx_n)) - mu\eta' \\ D_{x'} u_m & = (D_{x'} u)(1 - \eta(mx_n)). \end{cases}$$

Consequently we have that

$$\int_{\mathbb{R}^n_+} |Du_m - Du|^p dx \leq C \underbrace{\int_{\mathbb{R}^n_+} |\eta(mx_n)|^p |Du|^p dx}_{:=A_m} + \underbrace{C m^p \int_0^{2/m} \int_{\mathbb{R}^{n-1}} |u|^p dx' dt}_{:=B_m}$$

Now $A_m \to 0$ as $m \to 0$, as $\eta(mx_n) \neq 0$ if and only iff $0 \leq x_n \leq \frac{2}{m}$. To estimate B_m we see that

$$B_m \leq C m^p \left(\int_0^{2/m} t^{p-1} \, dt \right) \left(\int_0^{2/m} \int_{\mathbb{R}^{n-1}} |Du|^p dx' dx_n \right)$$

$$\leq C \int_0^{2/m} \int_{\mathbb{R}^{n-1}} |Du|^p dx_n \to 0 \text{ as } m \to \infty.$$

Finally using that $A_m, B_m \to 0$, we see that $Du_m \to Du$ in $L^p(\mathbb{R}^n_+)$ Since clearly $u_m \to u$ in $L^p(\mathbb{R}^n_+$ we conclude

$$u_m \to u \text{ in } W^{1,p}(\mathbb{R}^n_+).$$

But $w_m = 0$ if $0 < x_n < 1/m$. Therefore we mollify the w_m to produce functions $u_m \in C_c^\infty(\mathbb{R}^n)$ such that $u_m \to u$ in $W^{1,p}(\mathbb{R}^n_+)$. Hence $u \in W_0^{1,p}(\mathbb{R}^n_+)$.

7.4. Simple Hölder spaces

Let $\alpha \in (0,1)$ and let $(u_n)_{n \in \mathbb{N}} \subset C^{0,\alpha}(\bar{\Omega})$. We want to show that there is a $u \in C^{0,\alpha}(\bar{\Omega})$ such that $\lim_{n \to \infty} u_n = u$ with respect to the $C^{0,\alpha}$-norm. We know by completeness of $C^0(\bar{\Omega})$ that there exists a $u \in C^0(\bar{\Omega})$ with $\lim_{n \to \infty} u_n = u$ in the C^0 norm. Clearly we hope for u to be our $C^{0,\alpha}$ candidate as well.
Let us first show that in fact $u \in C^{0,\alpha}(\bar{\Omega})$. We then have for fixed $x \neq y$ that

$$\frac{|u(x) - u(y)|}{|x-y|^\alpha} \leq \frac{|u(x) - v_n(x)|}{|x-y|^\alpha} + \frac{|u_n(x) - u_n(y)|}{|x-y|^\alpha} + \frac{|u_n(y) - u(y)|}{|x-y|^\alpha} \tag{10}$$

taking the sup we can make the first and third term arbitrariy by choosing m large enough, as $u_n \to u$ uniformly. The second term can also be made arbitrarily small as u_n is Cauchy thus bounded, whence we conclude that

$$\frac{|u(x) - u(y)|}{|x - y|^\alpha} \leq M, \tag{11}$$

i.e. $u \in C^{0,\alpha}(\bar{\Omega})$. To conclude that the sequence also converges in the $C^{0,\alpha}$-norm we see that

$$\begin{aligned}\frac{|(u - u_n)(x) - (u - u_n)(y)|}{|x - y|^\alpha} &= \frac{|u(x) - u_n(x) - u(y) - u_n(y)|}{|x - y|^\alpha} \\ &= \frac{|\lim_{k\to\infty}(u_k(x) - u_n(x)) - \lim_{k\to\infty}(u_k(y) - u_n(y))|}{|x - y|^\alpha} \\ &= \lim_{k\to\infty} \frac{|(u_k - u_n)(x) - (u_k - u_n)(y)|}{|x - y|^\alpha}\end{aligned}$$

The last limit goes to zero as $(u_k - u_n) \in C^{0,\alpha}(\bar{\Omega})$

7.5. Extra: Properties of the characteristic function

This solution will be added shortly.