8.1. A Poincaré-like inequality on the unit ball.

Assume by contradiction that the statement is false. Then there exists a sequence $\{u_k\}_{k\in\mathbb{N}}$ with

$$\lambda\left(\{u_k(x)=0\}\right)\geq\alpha, \tag{1}$$

with $||u_k||_{L^2}=1$, but $||\nabla u_k||_{L^2}\to 0$ as $k\to\infty$. Since $H^1(B(0,1))$ is a Hilbert space we may assume WLOG that $u_k\rightharpoonup u$ weakly as $k\to\infty$. By the Rellich compactness theorem this implies that $u_k\to u$ in $L^2(B(0,1))$. As $\|u_k\|_{L^2}=1$ for all $k\in\mathbb{N}$ by L^2 convergence we have that $||u||_{L^2}=1$ as well. Moreover we have that

$$\|\nabla u\|_{L^2}\leq\liminf_{k\to\infty}\|\nabla u_k\|_{L^2}=0,$$

again by weak H^1-convergence. Therefore u must be constant almost everywhere $u=c:=1/\sqrt{\mathrm{vol}}(B(0,1))$. But then for any $k\in\mathbb{N}$. But then for any $k\in\mathbb{N}$, if for arbitrary $k\in\mathbb{N}$

$$S_k=\{x\in B(0,1)|u_k(x)=0\}, \tag{2}$$

we have that

$$||u-u_k||_{L^2(B(0,1))}\geq||u-u_k||_{L^2(S_k)}=\int_S|u(x)|^2\,dx\geq\alpha c^2.$$

In particular, as all sets S_k are non-null sets with $\lambda(S_k)>\alpha$, we conclude that u_k cannot converge to u on $L^2(S_k)$ hence also not on $L^2(B(0,1))$.

8.2. A variant of Hardy's inequality. NOTE: This exercise had a significant typo. As this complicated matters by quite a bit we present here a rather straightforward solution.
We do note that it suffices to prove the inequality for $u\in C_c^\infty(\mathbb{R}^n)$ as one can always argue by density of $C_c^\infty(\mathbb{R}^n)$ in $H^1(\mathbb{R}^n)$. Let now $\lambda>0$ arbitrary, and consider

$$0\leq\int_{\mathbb{R}^n}|\lambda\nabla u+\frac{xu(x)}{|x|^2}|^2\,dx \tag{3}$$
$$=\int_{\mathbb{R}^n}|\lambda|^2|\nabla u|^2\,dx+\int_{\mathbb{R}^n}\frac{|xu(x)|^2}{|x|^2}\,dx+2\lambda\int_{\mathbb{R}^n}\nabla u\cdot\frac{xu(x)}{|x|^2}\,dx \tag{4}$$

Now note that

$$\begin{aligned}\int_{\mathbb{R}^n}\nabla u\cdot\frac{xu(x)}{|x|^2}dx&=\int_{\mathbb{R}^n}\cdot\frac{x}{|x|^2}u\nabla udx\\&=\int_{\mathbb{R}^n}\cdot\frac{x}{2|x|^2}\nabla(u^2)dx,\end{aligned}$$

(note that the later inequality was what was asked to compute in **(c)**). Since

$$\nabla \cdot \frac{x}{|x|^2} = \frac{n-2}{|x|^2},$$

then we find with integration by parts that

$$\int_{\mathbb{R}^n} \nabla u \cdot \frac{xu(x)}{|x|^2} dx = -\frac{n-2}{2} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx.$$

Now going back to (3) we have that

$$0 \leq \lambda^2 |\nabla u|^2 \, dx + (1-(n-2)\lambda) \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx.$$

The above is a second degree polynomial in $\lambda > 0$ which attains its minimal value for

$$\lambda = \frac{n-2}{2} \frac{\int u^2/|x|^2 dx}{\int |\nabla u|^2 dx}.$$

Plugging this equation into the last inequality we find

$$\int_{\mathbb{R}^n} \frac{u^2}{|x|^2} dx \leq \underbrace{\left(\frac{2}{n-2}\right)^2}_{=C} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx. \tag{5}$$

8.3. Uniform bounds on functions in $W^{n,1}$

Let $u \in C_c^\infty(\mathbb{R}^n)$ and let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ be arbitrary. Then,

$$\begin{aligned} u(x_1, \ldots, x_n) &= \int_{-\infty}^{x_1} \frac{\partial u}{\partial x_1}(s_1, x_2, \ldots, x_n) \, ds_1 \\ &= \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \frac{\partial^2 u}{\partial x_2 \partial x_1}(s_1, s_2, x_2, \ldots, x_n) \, ds_2 \, ds_1 \\ &= \int_{-\infty}^{x_1} \ldots \int_{-\infty}^{x_n} \frac{\partial^n u}{\partial x_n \ldots \partial x_1}(s_1, \ldots, s_n) \, ds_n \ldots \, ds_1, \\ \Rightarrow \; |u(x)| &\leq \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \left| \frac{\partial^n u}{\partial x_n \ldots \partial x_1}(s_1, \ldots, s_n) \right| ds_n \ldots \, ds_1 \leq \|u\|_{W^{n,1}(\mathbb{R}^n)}. \end{aligned}$$

Since $x \in \mathbb{R}^n$ is arbitrary,

$$\forall u \in C_c^\infty(\mathbb{R}^n): \quad \|u\|_{L^\infty(\mathbb{R}^n)} \leq \|u\|_{W^{n,1}(\mathbb{R}^n)} \tag{6}$$

follows. The inequality (6) remains true for arbitrary $u \in W^{n,1}(\mathbb{R}^n)$ by density of $C_c^\infty(\mathbb{R}^n)$ in $W^{n,1}(\mathbb{R}^n)$. Indeed, given $u \in W^{n,1}(\mathbb{R}^n)$, let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $C_c^\infty(\mathbb{R}^n)$ such that $\|u_k - u\|_{W^{n,1}(\mathbb{R}^n)} \to 0$ as $k \to \infty$. Since inequality (6) implies $\|u_k - u_m\|_{L^\infty(\mathbb{R}^n)} \leq \|u_k - u_m\|_{W^{n,1}(\mathbb{R}^n)}$ the sequence $(u_k)_{k\in\mathbb{N}}$ is Cauchy in $L^\infty(\mathbb{R}^n)$ and hence convergent to some v in $L^\infty(\mathbb{R}^n)$. In particular, $u_k(x) \to v(x)$ converges pointwise for almost every $x \in \mathbb{R}^n$ as $k \to \infty$. Moreover, since $\|u_k - u\|_{L^n(\mathbb{R}^n)} \to 0$ implies pointwise convergence almost everywhere on a subsequence, $v = u$ almost everywhere follows by uniqueness of limits. Passing to the limit $k \to \infty$ in $\|u_k\|_{L^\infty(\mathbb{R}^n)} \leq \|u_k\|_{W^{n,1}(\mathbb{R}^n)}$ proves the claim.

8.4. Horizontal derivatives

Given $u \in H^2(\mathbb{R}^n_+) \cap H^1_0(\mathbb{R}^n_+)$ and $h \in \mathbb{R} \setminus \{0\}$, let $D_{h,i}u \colon \mathbb{R}^n_+ \to \mathbb{R}$ be given by

$$D_{h,i}u(x) = \frac{u(x + he_i) - u(x)}{h},$$

where $e_i = (0, \ldots, 0, 1, 0 \ldots, 0, 0) \in \mathbb{R}^n$ has the entry 1 at position $i \in \{1, \ldots, n-1\}$.

The translation by he_i is an isometry of $H^1(\mathbb{R}^n_+)$ and carries $C_c^\infty(\mathbb{R}^n_+)$ into itself, so it carries its closure $H^1_0(\mathbb{R}^n_+)$into itself. Therefore, $u \in H^1_0(\mathbb{R}^n_+)$ implies $D_{h,i}u \in H^1_0(\mathbb{R}^n_+)$.

According to Satz 8.3.1.iii) the assumption $u \in H^2(\mathbb{R}^n_+)$ implies

$$\exists C < \infty \quad \forall h \in \mathbb{R}^n \setminus \{0\} : \quad \|D_{h,i}u\|_{H^1} \leq C.$$

Hence, there exists a sequence $h_k \xrightarrow{k\to\infty} 0$ such that $D_{h_k,i}u$ converges weakly in $H^1(\mathbb{R}^n_+)$ to some $v \in H^1(\mathbb{R}^n_+)$ as $k \to \infty$. Since $H^1_0(\mathbb{R}^n_+)$ is a closed subspace of $H^1(\mathbb{R}^n_+)$, it is weakly closed. Therefore, $v \in H^1_0(\mathbb{R}^n_+)$. Moreover, for any $\varphi \in C_c^\infty(\mathbb{R}^n_+)$ there holds

$$\begin{aligned}
\int_{\mathbb{R}^n_+} v\varphi \, dx &= \lim_{k\to\infty} \int_{\mathbb{R}^n_+} \frac{u(x + h_k e_i) - u(x)}{h_k} \varphi(x) \, dx \\
&= \lim_{k\to\infty} \frac{1}{h_k} \Big(\int_{\mathbb{R}^n_+} u(x + h_k e_i)\varphi(x) \, dx - \int_{\mathbb{R}^n_+} u(x)\varphi(x) \, dx \Big) \\
&= \lim_{k\to\infty} \frac{1}{h_k} \Big(\int_{\mathbb{R}^n_+} u(y)\varphi(y - h_k e_i) \, dy - \int_{\mathbb{R}^n_+} u(x)\varphi(x) \, dx \Big) \\
&= -\lim_{k\to\infty} \int_{\mathbb{R}^n_+} u(x) \frac{\varphi(x) - \varphi(x - h_k e_i)}{h_k} \, dx \\
&= -\int_{\mathbb{R}^n_+} u \frac{\partial \varphi}{\partial x_i} \, dx.
\end{aligned}$$

By definition of weak derivative,

$$\frac{\partial u}{\partial x_i} = v \in H_0^1(\mathbb{R}^n_+)$$

and the claim follows.