9.1. Another density statement

(a) For $p < n$ we can prove the statement by taking $\psi_k = \psi(kx)$ with $\psi \in C_c^\infty(\mathbb{R}^n)$ as described in the hint. WLOG we have $\psi \equiv 1$ on $B_1(0)$ and $\psi_k \equiv 1$ on any neighbourhood of $x = 0$ for any $k \in \mathbb{N}$. After substitution of $y = kx$ we get

$$||\nabla\psi_k||_{L^p}^p = k^{p-n} \int_{\mathbb{R}^n} |\nabla\psi(y)|^p dy = Ck^{p-n} \to 0.$$

(b) For $p = n$ the construction above fails. Instead let us consider the sequence

$$\psi_k(x) = \begin{cases} 1, & \text{for} \quad \text{for } |x| \leq 1/(e^{e^{k+1}}) = r_k \\ \log(\log(1/|x|)) - k & \text{otherwise} \\ 0 & \text{for} \quad \text{for } |x| \geq 1/e^{e^k} = R_k \end{cases}$$

If we now take the derivative we have that

$$|\nabla\psi_k(x)|^n = \begin{cases} |x|^{-n} |\log(|x|)|^{-n}, & \text{for } r_k < |x| < R_k \\ 0, & \text{otherwise.} \end{cases}$$

If we now substitute $s = \log(1/r)$ we get the estimate

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla\psi_k|^n \, dx &\leq C \int_{r_k}^{R_k} \frac{r^{n-1}}{r^n |\log(r)|^n} dr = C \int_{\log(1/R_k)}^{\log(1/r_k)} \frac{ds}{s^n} \\ &= \frac{C(n)}{|\log(R)|^{n-1}} \Bigg|_{r_k}^{R_k} \to 0 \quad \text{as} \quad k \to \infty. \end{aligned}$$

After convoluting with a standard mollifer

$$\phi_\epsilon(x) = \epsilon^{-n} \phi(x/\epsilon),$$

where we have

$$\phi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} / I_n & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases} \tag{1}$$

we get that $\phi_\epsilon * \psi_k(x) \to \tilde{\psi}_k \in C_c^\infty(\mathbb{R}^n)$ in $W^{1,p}$, as required.

(c) This now follows with the constructions we have shown above when we set

$$u_k = (1 - \psi_k)u.$$

We have that that

$$u_k \to u \text{ pointwise}$$

and moreover that

$$\nabla u_k = \nabla u - u \underbrace{(\nabla \psi_k)}_{\to 0} - \underbrace{\psi_k}_{\to 0} \nabla u.$$

pointwise.
Noting that $||u_k||_{L^p} \leq 2||u||_{L^p}$ and $||\nabla u_k||_{L^p} \leq 2||\nabla u||_{L^p} + \underbrace{||u \nabla \psi_k||_{L^p}}_{\to 0 \text{ in } L^p}$ we conclude with dominated convergence that $u_k \to u$ in $W^{1,p}(\mathbb{R}^n)$.

(d) Here in the original sheet there was a small typo. Obviously as the ψ_k above are members of $C_c^\infty(\mathbb{R}^n \backslash \{0\})$ we have proven that $C_c^\infty(\mathbb{R}^n \backslash \{0\})$ lies dense in $W^{1,p}(\mathbb{R}^n)$, i.e. $W_0^{1,p}(\mathbb{R}^n \backslash \{0\}) = W^{1,p}(\mathbb{R}^n)$

9.2. Weak solutions to the Dirichlet problem are continuous.

(a) We have seen in class that a minimizer v for $E(v)$ exists and solves $-\Delta v = 0$ in the weak sense. Uniqueness now follows from taking an arbitrary $\phi \in H_0^1(\Omega)$ and defining $w := v + \phi$. Then an easy computation shows us that

$$E(w) = E(v + \phi) = E(v) + \int_\Omega \nabla v \nabla \phi dx + \frac{1}{2} \int_\Omega |\nabla \phi|^2 dx.$$

Under the assumption that v solves $-\Delta v = 0$ weakly we get after partial integration that

$$\begin{aligned} E(w) = E(v + \phi) &= E(v) + \int_\Omega \nabla v \cdot \nabla \phi dx + \frac{1}{2} \int_\Omega |\nabla \phi|^2 dx. \\ &= E(v) - \int_\Omega \underbrace{\Delta v}_{=0} \phi dx + \frac{1}{2} \int_\Omega |\nabla \phi|^2 dx \geq E(v). \end{aligned}$$

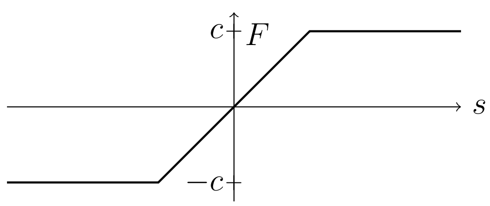
with equality if and only if $v = w$.

(b) Now consider $c := \|g\|_{L^\infty(\partial\Omega)}$, we claim that any weak solution $v \in H^1(\Omega)$ of $-\Delta v = 0$ with $v|_{\partial\Omega} = g$ satisfies

$$\|v\|_{L^\infty(\Omega)} \leq \|v|_{\partial\Omega}\|_{L^\infty(\partial\Omega)} \qquad (*)$$

and

$$F(s) = \begin{cases} c & \text{if } s > c, \\ s & \text{if } -c \leq s \leq c, \\ -c & \text{if } s < -c. \end{cases}$$



Then, $F \circ v \in H^1(\Omega)$ with the same trace g and $E(F \circ v) \leq E(v)$. By uniqueness of the minimiser, $F \circ v = v$. Therefore $|v| \leq c$ which proves the claim.

(c) Let u be harmonic (i.e. solve $-\Delta u$) in Ω and let $g = u|_{\partial\Omega} \in C^0(\partial\Omega)$. Let $(g_k)_{k\in\mathbb{N}}$ be a sequence in $C^\infty(\Omega)$ such that $g_k|_{\partial\Omega} \to g$ in $C^0(\partial\Omega)$ as $k \to \infty$. Let $v_k \in H^1_0(\Omega)$ be the weak solution of $-\Delta v_k = f_k$ where $f_k := \Delta g_k \in C^\infty(\Omega)$. By elliptic regularity, $v_k \in C^\infty(\Omega)$ and $v_k|_{\partial\Omega} = 0$. Thus, $u_k := v_k + g_k \in C^\infty(\Omega)$ satisfies $\Delta u_k = 0$ and $u_k|_{\partial\Omega} = g_k|_{\partial\Omega}$. Moreover, by $(*)$ $\|u_k - u\|_{L^\infty(\Omega)} \le \|g_k - g\|_{L^\infty(\partial\Omega)} \to 0$ as $k \to \infty$. As uniform limit of continuous functions, u is continuous in $\overline{\Omega}$.

9.3. Weak solutions to the biharmonic equation.

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary.

(a) The map $\langle\cdot,\cdot\rangle\colon \left(H^2(\Omega)\cap H^1_0(\Omega)\right)\times\left(H^2(\Omega)\cap H^1_0(\Omega)\right)\to\mathbb{R}$ given by

$$\langle u, v\rangle := \int_\Omega \Delta u \Delta v \, dx$$

is symmetric and bilinear by definition. Moreover, by the elliptic regularity estimate (Satz 9.1.1), there exists a constant $C < \infty$ such that for every $u \in H^2(\Omega)\cap H^1_0(\Omega)$

$$\langle u, u\rangle \le (u,u)_{H^2(\Omega)} = \|u\|^2_{H^2(\Omega)} \le C\|\Delta u\|^2_{L^2(\Omega)} = C\langle u, u\rangle.$$

In particular, $\langle u, u\rangle \ge 0$ and $\langle u, u\rangle = 0 \Leftrightarrow u = 0$; hence $\langle\cdot,\cdot\rangle$ defines a scalar product and $\langle\cdot,\cdot\rangle$ is equivalent to $(\cdot,\cdot)_{H^2(\Omega)}$.

(b) Since Ω is bounded, convergence in $H^2(\Omega)$ implies convergence in $H^1(\Omega)$. Since $H^1_0(\Omega)$ is closed in $H^1(\Omega)$, we obtain that $H^2(\Omega)\cap H^1_0(\Omega)$ is closed in $H^2(\Omega)$. Hence, $(H^2(\Omega)\cap H^1_0(\Omega), \langle\cdot,\cdot\rangle)$ is a Hilbert space.

(c) We first claim the following lemma:

Lemma. *Let $\Omega \subset \mathbb{R}^n$ open and bounded and define*

$$\Xi := \{u \in H^4(\Omega)\cap H^1_0(\Omega) \mid \Delta u \in H^1_0(\Omega)\}.$$

(i) Then the operator

$$\begin{aligned}\Delta^2\colon \Xi &\to L^2(\Omega)\\ u &\mapsto \Delta(\Delta u)\end{aligned}$$

is bijective.

(ii) For $f \in L^2(\Omega)$, then for $u \in \Xi$ satisfy $\Delta^2 u = f$ we have that

$$\forall\varphi\in\Xi:\quad \int_\Omega u\Delta^2\varphi\,dx = \int_\Omega f\varphi\,dx. \tag{2}$$

(iii) If $u, f \in L^2(\Omega)$ satisfy (2), then $u \in \Xi$

Proof. (i) Since the bilaplacian $\Delta^2 \colon \Xi \to L^2(\Omega)$ is linear, it suffices to prove $\ker(\Delta^2) = \{0\}$ to conclude injectivity. Let $u \in \Xi$ with $\Delta^2 u = 0$. By definition of Ξ, we have

$$v := \Delta u \in H^2(\Omega) \cap H_0^1(\Omega).$$

Moreover, $\Delta v = 0$ combined with the elliptic regularity estimate implies $v = 0$. Repeating the same argument for $\Delta u = 0$ yields $u = 0$ and proves $\ker(\Delta^2) = 0$.

To prove surjectivity, let $f \in L^2(\Omega)$ be given arbitrarily. Let $v \in H_0^1(\Omega)$ be the weak solution to $\Delta v = f$ in Ω. By elliptic regularity, $v \in H^2(\Omega)$. Let $u \in H_0^1(\Omega)$ be the weak solution to $\Delta u = v$. Then, by elliptic regularity, $u \in H^4(\Omega)$. Consequently, $u \in \Xi$. Since $\Delta^2 u = f$ by construction, surjectivity of $\Delta^2 \colon \Xi \to L^2(\Omega)$ follows.

(ii) Given $f \in L^2(\Omega)$, let $u \in \Xi$ satisfy $\Delta^2 u = f$. Let $\varphi \in \Xi$ be arbitrary. Then, $\nabla\Delta\varphi \in L^2(\Omega)$. Since $u \in H_0^1(\Omega)$, the trace theorem (Satz 8.4.3) implies that $u|_{\partial\Omega} \in L^2(\partial\Omega)$ is well-defined and vanishes according to Korollar 8.4.3. Analogously, since $\Delta\varphi \in H_0^1(\Omega)$ by assumption, $(\Delta\varphi)|_{\partial\Omega} = 0$. Hence, we may integrate by parts twice with vanishing boundary terms to obtain

$$\int_\Omega u\Delta^2\varphi\,dx = -\int_\Omega \nabla u \cdot \nabla\Delta\varphi\,dx = \int_\Omega \Delta u \Delta\varphi\,dx. \qquad (*)$$

Since the right hand side of $(*)$ is symmetric in u and φ we may switch the roles of $u, \varphi \in \Xi$ to also obtain

$$\int_\Omega \varphi\Delta^2 u\,dx = \int_\Omega \Delta u\Delta\varphi\,dx = \int_\Omega u\Delta^2\varphi\,dx.$$

Since $\varphi \in \Xi$ is arbitrary, the claim follows by substituting $\Delta^2 u = f$.

(iii) Given $f \in L^2(\Omega)$, let $u \in L^2(\Omega)$ satisfy

$$\forall\varphi \in \Xi: \quad \int_\Omega u\Delta^2\varphi\,dx = \int_\Omega f\varphi\,dx. \qquad (\dagger)$$

According to part (c), there exists $v \in \Xi$ such that $\Delta^2 v = f$. Moreover, by part (c)

$$\forall\varphi \in \Xi: \quad \int_\Omega v\Delta^2\varphi\,dx = \int_\Omega f\varphi\,dx.$$

Therefore, using again bijectivity of $\Delta^2 \colon \Xi \to L^2(\Omega)$ as shown in (c), we have

$$\forall\varphi \in \Xi: \quad \int_\Omega (u-v)\Delta^2\varphi\,dx = 0 \quad \overset{(c)}{\Longleftrightarrow} \quad \forall\psi \in L^2(\Omega): \quad \int_\Omega (u-v)\psi\,dx = 0.$$

Hence $u - v = 0$ in $L^2(\Omega)$. Therefore, $u = v \in \Xi$ as claimed.

□

Now we return to our original proof: Let $f \in L^2(\Omega)$. Then the map $H^2(\Omega) \cap H_0^1(\Omega) \to \mathbb{R}$ given by $v \mapsto \int_\Omega fv\,dx$ is a continuous linear functional. By part (b) we may apply the Riesz representation theorem to conclude that there exists a unique $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying

$$\forall v \in H^2(\Omega) \cap H_0^1(\Omega): \quad \langle u, v\rangle = \int_\Omega fv\,dx.$$

In particular, for any $v \in \Xi := \{u \in H^4(\Omega) \cap H_0^1(\Omega) \mid \Delta u \in H_0^1(\Omega)\}$,

$$\int_\Omega u\Delta^2 v\,dx = \int_\Omega \Delta u \Delta v = \int fv\,dx.$$

Hence, $u \in \Xi$ according to part (iii) of the previous lemma and

$$\int_\Omega (\Delta^2 u)v\,dx = \int_\Omega u\Delta^2 v\,dx = \int fv\,dx$$

for any $v \in C_c^\infty(\Omega)$ which implies $\Delta^2 u = f$.

9.4. Weak solution to a semilinear equation.

We follow the proof of elliptic regularity. To start off, integrating the equation against a testfunction $v \in H_0^1(\mathbb{R}^N)$ we get

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v dx = \int_{\mathbb{R}^n} (f - c(u))v\,dx. \tag{3}$$

Now $|h| > 0$ small and an index $n \in \{1, ..., N\}$ and we set $v = -D_n^{-h}(D_n^h)u$ where

$$D_k^h u(x) := \frac{u(x + he^n) - u(x)}{h}$$

is the standard difference quotient. Recall that for D_n^{-h} we have the following identities:

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi D_n^{-h}\psi\ dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} D_n^h\phi\psi\ dx,$$

and moreover that

$$D_n^h \partial_{x_i}\phi = \partial_{x_i}(D_n^h\phi)$$

We can write the l.h.s of (3) as

$$\begin{aligned}
\int_{\mathbb{R}^N} \nabla u \cdot \nabla v\,dx &= \int_{\mathbb{R}^N} \nabla u \cdot \nabla(D_n^{-h} D_n^h u)dx \\
&= \int_{\mathbb{R}^N} \nabla u \cdot D_n^{-h}(\nabla D_n^h u)\,dx \\
&= \int_{\mathbb{R}^N} \nabla u \cdot D_n^{-h} D_n^h(\nabla u)\,dx \\
&= \int_{\mathbb{R}^N} D_n^h \nabla u \cdot D_n^h(\nabla u)\,dx \\
&= ||D_n^h \nabla u||_{L^2}^2.
\end{aligned}$$

Now the r.h.s of (3) can be estimated as

$$\left|\int_{\mathbb{R}^n}(f-c(u))vdx\right| = \left|\int_{\mathbb{R}^n} -fD_n^{-h}D_n^h u + c(u)D_n^{-h}D_n^h udx\right| \leq \int_{\mathbb{R}^N} (|f|+|c(u)|)\,|D_n^{-h}D_n^h u|\;dx.$$

Now note that by the fundamental theorem of calculus we have $c(0)=0$ and $c'\in L^\infty$ and hence we find

$$|c(u(x))| = \left|\int_0^{u(x)} c'(t)\;dt\right| \leq ||c'||_{L^\infty}|u(x)|,$$

recall that as u has compact support the integral of 0 to $u(x)$ Also note that from that this we have according to remark $(R.28)$ the following inequality

$$\int_{\mathbb{R}^N} |D_n^{-h}D_n^h u|^2\;dx \leq C_1\int_{\mathbb{R}^N} |\nabla D_n^h u|^2\;dx \leq C_2\int_{\mathbb{R}^n} |\nabla u|^2\;dx.$$

Taking this all together we find

$$\begin{aligned}\left|\int_{\mathbb{R}^n}(f-c(u))v\;dx\right| &\leq \int_{\mathbb{R}^N} \nabla u\cdot\nabla(D_n^{-h}D_n^h u)dx\\ &\leq \int_{\mathbb{R}^N} (|f|+||c'||_{L^\infty}|u|)\,|D_n^{-h}D_n^h u|\;dx\\ &\leq \frac{1}{2}\int_{\mathbb{R}^N} (|f|+||c'||_\infty|u|)\;dx + \frac{C_2}{2}\int_{\mathbb{R}^N}|\nabla u|^2\;dx\\ &\leq C_3(||f||_{L^2}^2+||u||_{L^2}^2+||\nabla u||_{L^2}^2)\end{aligned}$$

for some $C_3>0$ We conclude

$$\begin{aligned}||D_n^h\partial_{x_i}u||_{L^2}^2 &\leq ||D_n^h\nabla u||_2^2\\ &= \left|\int_{\mathbb{R}^n}(f-c(u))v\;dx\right|\\ &\leq C_3(||f||_{L^2}^2+||u||_{L^2}^2+||\nabla u||_{L^2}^2)\end{aligned}$$

We conclude that

$$||D_n^h\partial_{x_i}u||_{L^2}^2 \leq ||D_n^h\nabla u||_{L^2}^2 \leq C_3(||f||_{L^2}^2+||u||_{L^2}^2+||\nabla u||_{L^2}^2)$$

for small $|h|>0$, and $i\in\{1,...,N\}$ arbitrary. We conclude by elliptic regularity that $\partial_{x_i}u\in H^1(\mathbb{R}^N)$ and

$$||\nabla\partial_{x_i}u||_{L^2}^2 \leq ||D_n^h\nabla u||_2^2 \leq C_3\left(||f||_2^2+||u||_2^2+||\nabla u||_2^2\right),$$

so by elliptic regularity we have that $u\in H^2(\mathbb{R}^N)$.

9.5. RECAP 1: Fundamental solution to Poisson's equation on $\mathbb{R}^n$

(a) Making the radial symmetry assumption of $u(x) = v(r)$, with $r = (x_1^2 + ... + x_n^2)^{1/2}$ we see that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

We therefore have

$$\partial_{x_i} u = v'(r)\frac{x_i}{r} \quad \text{and} \quad \partial_{x_i}^2 u = v''(r)\frac{x_i^2}{r^2} + v'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right). \tag{4}$$

We see then that

$$\Delta u = v''(r) + \frac{n-1}{r}v'(r),$$

so $\Delta u = 0$ if and only if

$$v'' + \frac{n-1}{r}v' = 0.$$

If $v' \neq 0$ then we deduce

$$\log(|v'|)' = \frac{v''}{v'} = \frac{1-n}{r},$$

and hence $v'(r) = \frac{a}{r^{n-1}}$. For $r > 0$ we can therefore solve this ODE as

$$v(r) = \begin{cases} b\log(r) + c & (n = 2) \\ \frac{b}{r^{n-2}} + c & (n \geq 3). \end{cases}$$

(b) We have

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,dy = \int_{\mathbb{R}^n} \Phi(y)f(x-y)\,dy$$

Therefore we have for the difference quotient

$$D_i^h u = \frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^N} \Phi(y)\left[\frac{f(x+he_i-y) - f(x-y)}{h}\right]dy.$$

but

$$\frac{f(x+he_i-y) - f(x-y)}{h} \to \partial_{x_i} f(x-y)$$

uniformily on $\mathbb{R}^n$ as $h \to 0$, and thus taking the limit on both sides we find

$$\partial_{x_i} u = \int_{\mathbb{R}^n} \Phi(y)\partial_{x_i} f(x-y)\,dy$$

for $i = 1, ..., n$. Similarly we see for

$$\partial_{x_i}\partial_{x_j} u = \int_{\mathbb{R}^n} \Phi(y)\partial_{x_i}\partial_{x_j} f(x-y)\,dy$$

Now as the expression on the r.h.s is continuous in x we see that $u \in C^2(\mathbb{R}^n)$.
To show that $-\Delta u(x) = f(x)$ we need to take into account that Φ blows up at 0 we need to isolate this singularity. Let $B(0,\epsilon)$ be a ball centered at $x = 0$ with radius $\epsilon > 0$. Then

$$\Delta u(x) = \underbrace{\int_{B(0,\epsilon)} \Phi(y)\Delta_x f(x-y)dy}_{:=I_\epsilon} + \underbrace{\int_{\mathbb{R}^n\setminus B(0,\epsilon)} \Phi(y)\Delta_x f(x-y)\, dy}_{:=J_\epsilon}.$$

We then note that

$$|I_\epsilon| \leq C||D^2 f||_{L^\infty} \int_{B(0,\epsilon)} |\Phi(y)|\, dy = \begin{cases} C\epsilon^2|\log(\epsilon)| & (n=2) \\ C\epsilon^2 & (n \geq 3). \end{cases}$$

And integration by parts yields for J_ϵ the following

$$J_\epsilon = \underbrace{-\int_{\mathbb{R}^n\setminus B(0,\epsilon)} D\Phi(y)\cdot D_y f(x-y)\, dy}_{:=K_\epsilon} + \underbrace{\int_{\partial B(0,\epsilon)} \Phi(y)\frac{\partial f}{\partial \nu}(x-y)dS(y)}_{:=L_\epsilon}$$

where ν is the inward pointing unit normal on $\partial B(0,\epsilon)$, where we also implicitly used that

$$\int_{\mathbb{R}^n\setminus B(0,\epsilon)} \Phi(y)\Delta_x f(x-y)\, dy = -\int_{\mathbb{R}^n\setminus B(0,\epsilon)} \Phi(y)\Delta_y f(x-y)\, dy$$

We now also see that

$$|L_\epsilon| \leq ||Df||_{L^\infty} \int_{\partial B(0,\epsilon)} |\Phi(y)|\, dS(y) \leq \begin{cases} C\epsilon|\log(\epsilon)| & (n=2) \\ C\epsilon & (n \geq 3) \end{cases}$$

We note that I_ϵ and L_ϵ converge uniformily to 0 as $\epsilon \searrow 0$ both for the cases for $n = 2$ and $n \geq 3$. Finally we need to estimate K_ϵ and hope that it converges to $-f(x)$. We see that

$$\begin{aligned} K_\epsilon &= \int_{\mathbb{R}^n\setminus B(0,\epsilon)} \underbrace{\Delta\Phi(y)}_{=0} f(x-y)\, dy - \int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \nu} f(x-y)\, dS(y) \\ &= -\int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \nu} f(x-y)\, dS(y), \end{aligned}$$

where the first term vanishes as Φ is the spherically symmetric solution that solves $-\Delta\Phi(y) = 0$ away from the origin, by the previous exercise. Now we note that $D\Phi(y) = -\frac{1}{n\alpha(n)}\frac{y}{|y|^n}$ for $y \neq 0$ and $\nu = -\frac{y}{|y|} = -\frac{y}{\epsilon}$ on $\partial B(0,\epsilon)$. Consequently $\frac{\partial \Phi}{\partial \nu} = \nu \cdot D\Phi(y) = \frac{1}{n\alpha(n)\epsilon^{n-1}}$ on $\partial B(0,\epsilon)$, where $\alpha(n)$ is the solid angle in n dimensions. Note $n\alpha(n)\epsilon^{n-1}$ is therefore the surface area of the ball $\partial B(0,\epsilon)$. We therefore see that

$$\begin{aligned} K_\epsilon &= -\frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(0,\epsilon)} f(x-y)\, dS(y) \\ &= -⨍_{\partial B(x,\epsilon)} f(y)\, dS(y) \to -f(x) \end{aligned}$$

as $\epsilon \searrow 0$, where we performed a translation. Combining now all these results we see that

$$\Delta u(x) = -\Delta(\Phi * f) = \lim_{\epsilon \searrow 0} \underbrace{I_\epsilon}_{\to 0} + \underbrace{L_\epsilon}_{\to 0} + \underbrace{K_\epsilon}_{\to -f(x)} = -f(x)$$

uniformily as required.

9.6. RECAP 2: Fundamental solution to Poisson's equation on the unit ball.

(a) Choosing $B(x,\epsilon)$ for $\epsilon > 0$ small and let V_ϵ be as in the hint. We then apply Green's formula to $u(y)$ and $\Phi(y-x)$ on V_ϵ and see that

$$\begin{aligned}\int_{V_\epsilon} & u(y)\Delta\Phi(y-x) - \Phi(y-x)\Delta u(y)\; dy \\ &= \int_{\partial V_\epsilon} u(y)\frac{\partial \Phi}{\partial \nu}(y-x) - \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\; dS(y)\end{aligned}$$

where now ν is the outer unit normal to ∂V_ϵ. Recall next that $\Delta\Phi(x-y) = 0$ for $x \neq y$. Then observe also that

$$\left| \int_{\partial B(x,\epsilon)} \Phi(y-x)\frac{\partial u}{\partial \nu}(y) dS(y) \right| \leq C\epsilon^{n-1} \max_{\partial B(0,\epsilon)} |\Phi|,$$

which goes to 0 uniformily in the limit $\epsilon \searrow 0$. Furthermore the previous calculations from exercise 5 show that

$$\int_{\partial B(x,\epsilon)} u(y)\frac{\partial \Phi}{\partial \nu}(y-x) dS(y) = \fint_{\partial B(x,\epsilon)} u(y) dS(y) \to u(x).$$

as $\epsilon \searrow 0$. Hence sending $\epsilon \to 0$ yields the formula

$$u(x) = \int_{\partial \Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y) - u(y)\frac{\partial \Phi}{\partial \nu}(y-x)\; dS(y) \tag{5}$$
$$- \int_\Omega \Phi(y-x)\Delta u(y), \tag{6}$$

as desired.

(b) To start off we apply now Green's formula to the corrector function $\phi^x(y)$. We see that

$$\begin{aligned} -\int_\Omega \phi^x(y)\Delta u(y)\; dy &= \int_{\partial\Omega} u(y)\frac{\partial \phi^x}{\partial \nu}(y) - \phi^x(y)\frac{\partial u}{\partial \nu}(y)\; dS(y) && (7)\\ &= \int_{\partial\Omega} u(y)\frac{\partial \phi^x}{\partial \nu}(y) - \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\; dS(y). && (8)\end{aligned}$$

Now defining $G(x,y) = \Phi(y-x) - \phi^x(y)$ and adding (7) to (5), we find

$$u(x) = -\int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu}(x,y)dS(y) + \int_\Omega G(x,y)\Delta u(y)dy \quad \text{for} \quad x \in \Omega,$$

Thus if $u \in C^2(\bar{\Omega})$ solves

$$-\Delta u = f \quad \text{on} \quad \Omega \tag{9}$$
$$u = g \quad \text{on} \quad \partial\Omega, \tag{10}$$

we get by plugging in the above that

$$u(x) = -\int_{\partial\Omega} u(y)\frac{\partial G}{\partial \nu}(x,y)dS(y) + \int_\Omega G(x,y)\Delta u(y)dy \quad \text{for} \quad x \in \Omega, \tag{11}$$

as required, with $\frac{\partial G}{\partial \nu}(x,y) := D_y G(x,y) \cdot \nu(y)$.

(c) As $\Phi(y-x) \propto |x-y|^{-(n-2)}$ on $\partial B(0,1)$ we note that $\phi^x(y)$ is harmonic for $y \neq \tilde{x}$ (i.e. $-\Delta\phi^x(y) = 0$. Note also that for $y \in \partial B(0,1)$

$$\begin{aligned} |x|^2|y-\tilde{x}|^2 &= |x|^2\left(|y|^2 - \frac{2x\cdot y}{|x|^2} + \frac{1}{|x|^2}\right) \\ &= |x|^2 - 2y\cdot x + 1 = |x-y|^2, \end{aligned}$$

as $|y| = 1$. Therefore we see that $|x||y-\tilde{x}|^{-(n-2)} = |x-y|^{-(n-2)}$ and we conclude (using the definition of Φ from the previous exercise) that

$$\begin{aligned} \phi^x(y) &= \Phi(|x|(y-\tilde{x})) \\ &= \frac{1}{n(n-2)\alpha(n)}\frac{1}{(|x||y-\tilde{x}|)^{n-2}} \\ &= \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x-y|^{n-2}} = \Phi(y-x), \end{aligned}$$

so $\phi^x(\cdot)$ fullfills the boundary conditions.

(d) From the previous exercise and the definition of the greenfunction we see that the Green's function for the unit ball is given by

$$G(x,y) := \Phi(y-x) - \Phi(|x|(y-\tilde{x})) \tag{12}$$

for $x, y \in B(0,1)$ with $x \neq y$. Assume now that $u \in C^2(\bar{\Omega})$ solves

$$-\Delta u = 0 \quad \text{on} \ \Omega \tag{13}$$
$$u = g \quad \text{on} \ \partial\Omega, \tag{14}$$

then from (11) we get

$$u(x) = -\int_{\partial B(0,1)} g(y) \frac{\partial G}{\partial \nu}(x, y)\, dS(y). \tag{15}$$

We then have from (12) that

$$\partial_{y_i} G(x, y) = \partial_{y_i} \Phi(y - x) - \partial_{y_i} \Phi(|x|(y - \tilde{x})).$$

but

$$\partial_{y_i} \Phi(y - x) = \frac{1}{n\alpha(n)} \frac{x_i - y_i}{|x - y|^n}$$

and furthermore we have

$$\begin{aligned}
\partial_{y_i} \Phi(|x|(y - \tilde{x})) &= \sum_{i=1}^{n} y_i \partial_{y_i} G(x, y) \\
&= -\frac{1}{n\alpha(n)} \frac{1}{|x - y|^n} \sum_{i=1}^{n} y_i((y_i - x_i) - y_i|x|^2 + x_i) \\
&= -\frac{1}{n\alpha(n)} \frac{1 - |x|^2}{|x - y|^n}
\end{aligned}$$

plugging this back into (15) we get the representation formula

$$u(x) = \frac{1 - |x|^2}{n\alpha(n)} \int_{\partial B(0,1)} \frac{g(y)}{|x - y|^n}\, dS(y). \tag{16}$$

For the same boundary value problem on $B(0, r)$ we can rescale the equation to

$$u(x) = \int_{\partial B(0,r)} K(x, y) g(y) dy, \tag{17}$$

where

$$K(x, y) := \frac{r^2 - |x|^2}{n\alpha(n) r} \frac{1}{|x - y|^n} \tag{18}$$

with $x \in B(0, r)$ and $y \in \partial B(0, r)$