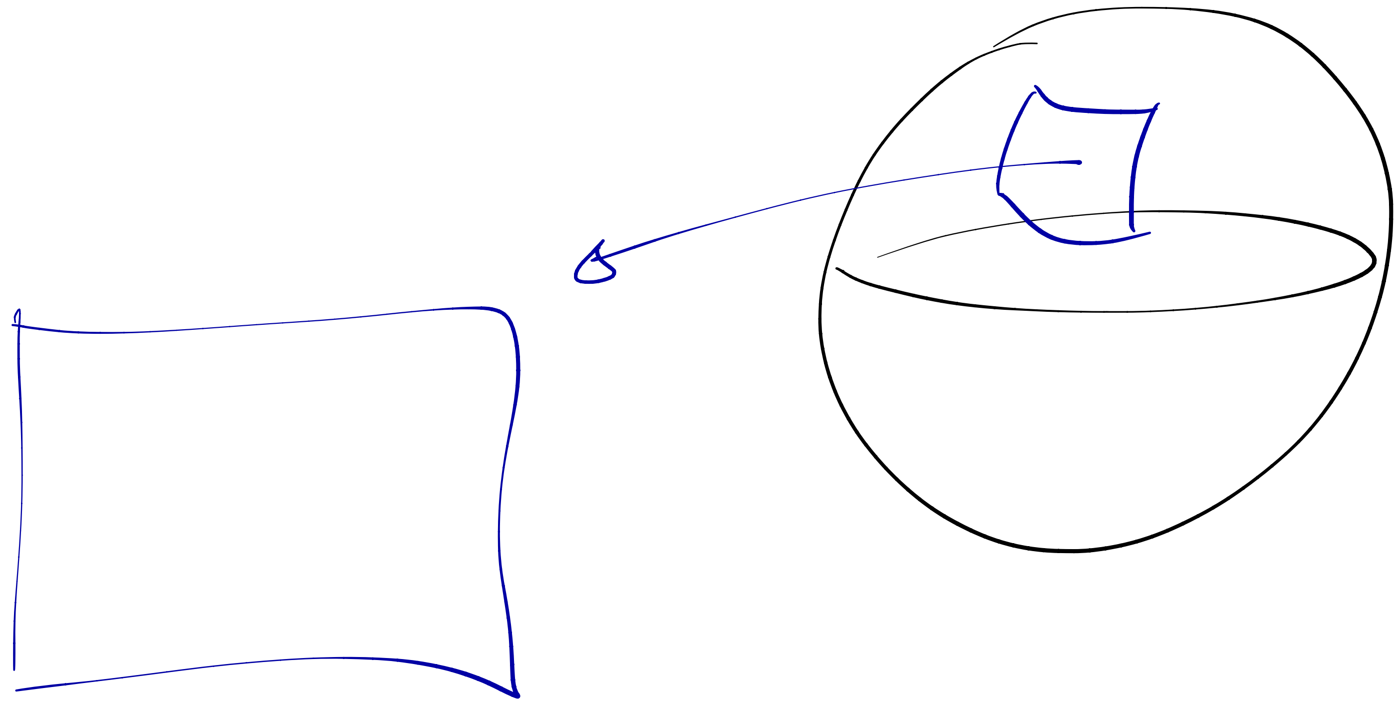
Euclidean space (300 BC) $\cong$ $\boxed{\begin{array}{l}\mathbb{R}^3, \|\ \| / \langle\ ,\ \rangle \\ \|x\| = \sqrt{(x^1)^2+(x^2)^2+(x^3)^2}\end{array}}$

Axioms

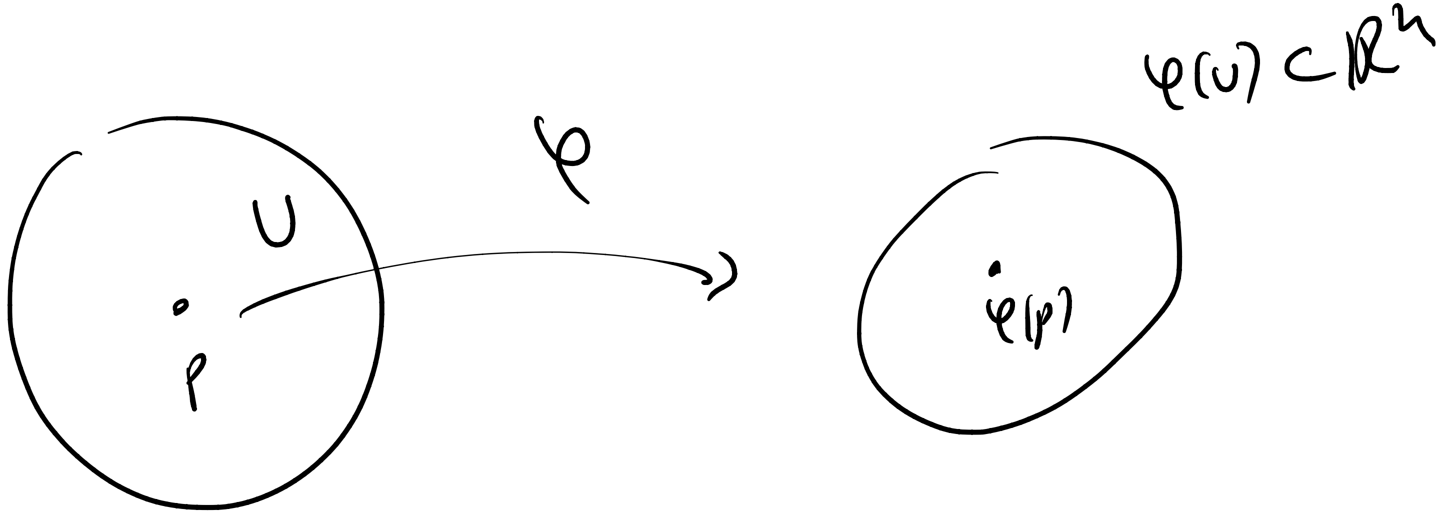
Descartes 1630's

Differential Geometry before XIX c. (DGI)

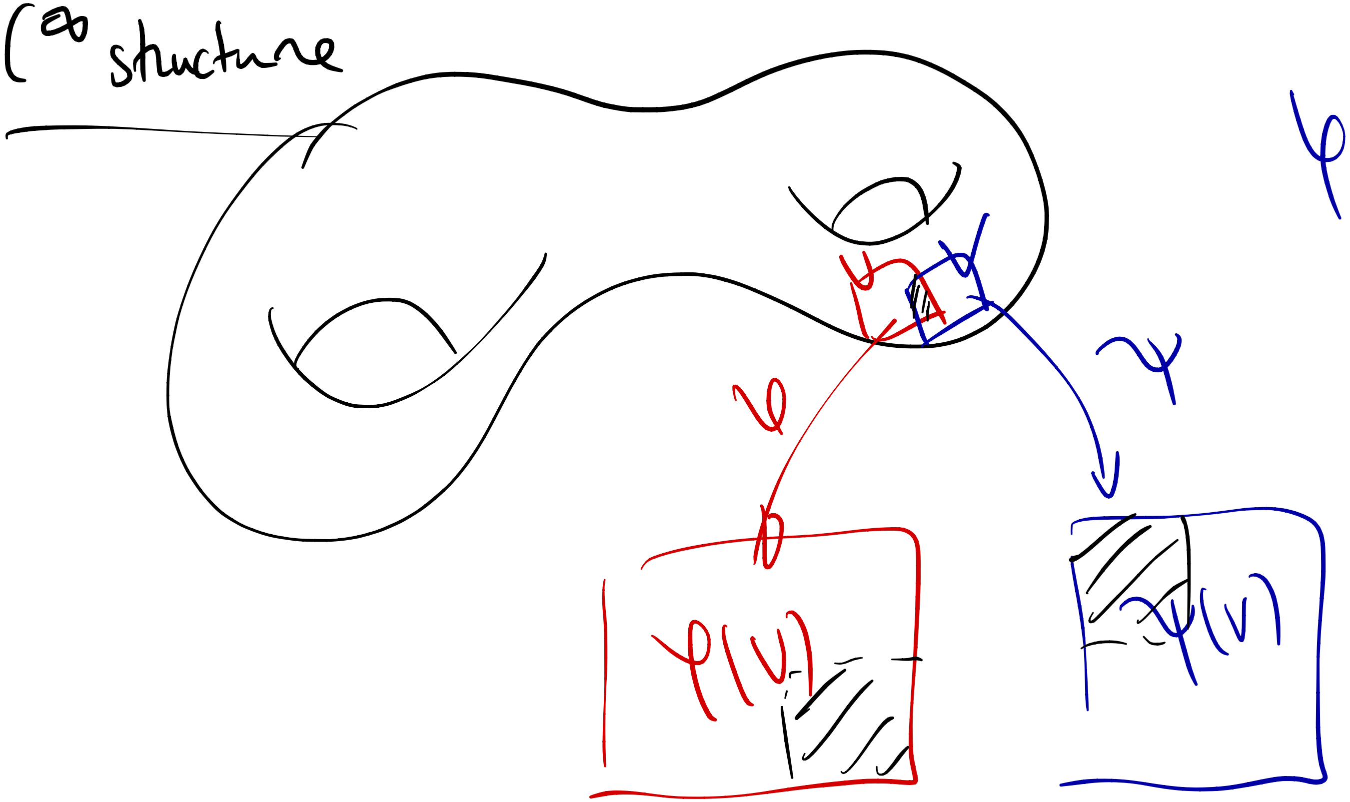


Differentiable mfld (Ch 8 Urs Lang's Diff Geom (1) lecture notes)

(8.1) Topological mfld M, of dim m, is Hausdorff top. space with countable basis and the property $\forall p \in M$ $\exists U \ni p$ open nbhd $\exists \varphi : U \longrightarrow \varphi(U) \subset \mathbb{R}^m$



- chart / atlas
- C^∞ structure



$\varphi \circ \psi^{-1} \in C^\infty$

$\forall$ 2 charts in the atlas

M, N endowed with C^∞ structures

$F : M \to N$ is C^∞ (smooth)

$\psi \circ F \circ \varphi^{-1}$ are C^∞

Tangent bundle → vectors on chart / ~
→ derivations
→

Goal of first lectures → Introduce metric g

(M, g) → Riemannian manifold

Vector bundles → Ch 10 in Lang's DG1 notes

Ch 10. Vector bundles, vector fields, and flows

10.1 Def A (real C^∞) vector bundle with fiber dimension k is a triple (π, E, M) s.t E (total space) and M (base space) are C^∞ mflds, $\pi : E \to M$ is C^∞ (projection) and:

(1) $\forall p \in M$, the fiber $E_p := \pi^{-1}\{p\}$ has the structure of a k-dim (real) vector space

(2) $\forall q \in M$ $\exists$ open nbhd $U \subset M$ of q and a C^∞ diffeom. $\Psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ (bundle chart)

s.t $\Psi|_{E_p} : E_p \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism $\forall p \in U$

Examples 1. $\forall M$ trivial $\mathbb{R}^k$-bundle over M

$$\pi : M \times \mathbb{R}^k \longrightarrow M \qquad \pi(p,\xi) = p$$

(id on $M \times \mathbb{R}^k$ is global bundle chart)

2. Tangent bundle $\pi : TM \longrightarrow M \qquad TM = \bigcup_{p \in M} TM_p$

$$[\varphi,\xi]_p \longmapsto p$$

(φ, U) chart of M

$$\Rightarrow \quad \pi^{-1}(U) = TU \xrightarrow{T\varphi} \varphi(U) \times \mathbb{R}^m \longrightarrow U \times \mathbb{R}^m$$

$$[\varphi,\xi]_p \longmapsto (\varphi(p), \xi) \longmapsto (p, \xi)$$

3. $M = \mathbb{R}P^m = \{ [x] = \{\pm x\} \mid x \in S^m \subset \mathbb{R}^{m+1} \}$

canonical line bundle

$$E := \{ ([x], v) \mid [x] \in \mathbb{R}P^m , v \in \mathbb{R}x \}$$

$$\pi([x], v) = [x]$$

A vec bundle with fiber dim k, aka <u>k-plane bundle</u>, is <u>trivial</u> if $\exists$ global bundle chart

$$\psi : E \longrightarrow M \times \mathbb{R}^k$$

$\pi : E \to M$ v.b

A C^∞ map $s : M \to E$ with $\pi \circ s = id_M$ (i.e. $s(p) \in E_p \ \forall p \in M$) is called a <u>section</u> of E

We denote $\Gamma(E)$ the set of all sections of E

Examples 1. zero section $s(p) = 0 \in E_p \quad \forall p \in M$

(check it is smooth!)

2. $\Gamma(TM) = \{ C^\infty$ v.f. on $M \}$

10.3. Prop. A k-plane bundle $\pi : E \to M$ is trivial iff $\exists$ k everywhere lin. ind. sections

pf $s_1, \dots, s_k \in \Gamma(E)$ s.t. $s_1(p), \dots, s_k(p) \in E_p$ lin. indep $\forall p \in M$

Define $\psi : E \to M \times \mathbb{R}^k$

$$\sum_{i=1}^{k} \xi^i s_i(p) \longmapsto (p, \xi)$$

[Check this is smooth]

Conversely, if $\Psi : E \to M \times \mathbb{R}^k$ global bundle chart

then we can define

$$s_i : M \to E$$

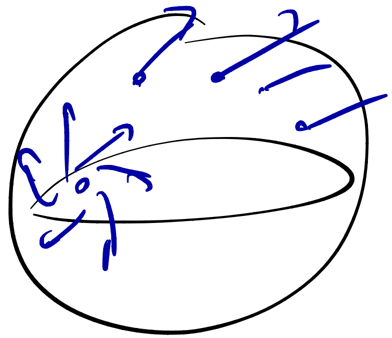
$$s_i(p) = \Psi^{-1}(p, e_i) \quad \text{for } i = 1, \dots, k,$$

e_i is i-th element of can. basis of $\mathbb{R}^k$

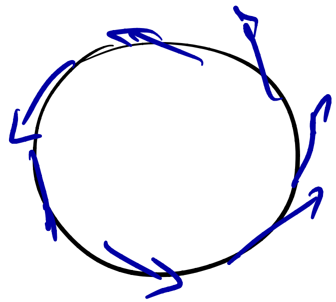
Hairy ball thm on even dimensional spheres S^m $m = 2k$

every vector field must vanish somewhere S^2

<u>Corollary</u> TS^{2k} is nontrivial $\forall k \geq 1$



Kervaire / Bott Milnor 1958 : S^1, S^3, S^7 are the only spheres with trivial tangent bundle



For $S^3 \subset \mathbb{R}^4$ $\quad \{(p^1, p^2, p^3, p^4) : |p|_{\text{Eucl}} = 1\}$

$$S_1(p) = (-p^2, p^1, -p^4, p^3)$$

$$S_2(p) = (-p^3, p^4, p^1, -p^2)$$

$$S_3(p) = (-p^4, -p^3, p^2, p^1)$$

<u>Cotangent bundle</u>

$:= \{\lambda : TM_p \to \mathbb{R} \text{ linear}\}$

<u>dual space</u> of TM_p

$$TM^* = \bigcup_{p \in M} TM_p^*$$

$$\begin{array}{c} TM_p^* \\ \downarrow \end{array}$$

$$\pi : TM^* \longrightarrow M \qquad \pi(\lambda) = p$$

(φ, U) chart of M

$$\Longrightarrow \quad \psi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^m$$

$$\underset{\substack{\cap \\ TM_p^*}}{\lambda} \longmapsto \Big(p, \underbrace{\sum_{i=1}^{m} \lambda\Big(\frac{\partial}{\partial \varphi^i}\Big|_p\Big) e_i}_{\mathbb{R}^m}\Big)$$

is <u>bundle chart</u>!

The differentials $d\varphi^1_p, \ldots, d\varphi^m_p : TM_p \longrightarrow \mathbb{R}$

constitute the basis of TM_p^* dual to

$\frac{\partial}{\partial \varphi^1}\Big|_p, \ldots, \frac{\partial}{\partial \varphi^m}\Big|_p$

$$d\varphi^i_p\Big(\frac{\partial}{\partial \varphi^j}\Big|_p\Big) = \frac{\partial \varphi^i}{\partial \varphi^j}(p) = \delta^i_j$$

A section $\omega \in \Gamma(TM^*)$ is a covector field or 1-form

w.r.t. a chart (φ, U) of M, ω has a unique rep.

$$\omega_p = \sum_{i=1}^{m} \underbrace{\omega_i(p)}_{\text{coeff.}} \, d\varphi^i_p$$

for C^∞ fcn's $\omega_i : U \longrightarrow \mathbb{R}$ $\qquad \omega_i(p) = \omega_p\left(\left.\frac{\partial}{\partial \varphi^i}\right|_p\right)$

Briefly $\quad \omega|_U = \sum_{i=1}^{m} \omega_i \, d\varphi^i$

Pull-back bundle or induced bundle:

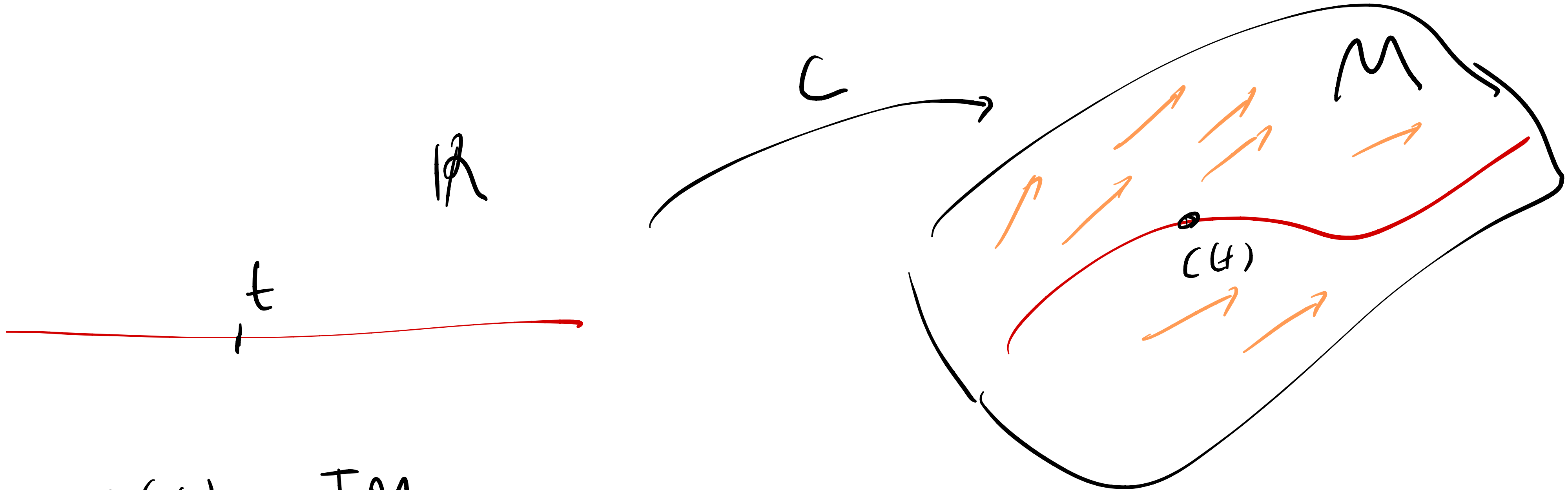
$$\begin{array}{ccc} & & E' \\ & & \downarrow \pi' \\ M & \xrightarrow[C^\infty]{F} & M' \end{array}$$

k-plane bundle

$$F^*E' := \{ (p, v) \in M \times E' : F(p) = \pi'(v), \text{ i.e } v \in E'_{F(p)} \}$$

$$\pi : F^*E' \longrightarrow M$$
$$(p,v) \longmapsto p$$

A section $s \in \Gamma(F^*TM')$ is a vector field along F



$$s(t) \in TM_{c(t)}$$

Riemannian mflds

M m-dim smooth (C^∞) mfld

TM, TM^* tangent cotangent bundles

$\Gamma(TM)$ vect. field $\quad$ $\Gamma(TM^*)$ 1-form

Given $r, s \geq 0$ integers $\quad$ (r,s)-tensor bundle

$$\left[\, T_{r,s}M = \underbrace{TM \otimes \cdots \otimes TM}_{r} \otimes \underbrace{TM^* \otimes \cdots \otimes TM^*}_{s} \,\right]$$

$$\boxed{T_{r,s}M_p = \underbrace{TM_p \otimes \cdots \otimes TM_p}_{r} \otimes \underbrace{TM_p^* \otimes - \otimes TM_p^*}_{s}}$$

this is equivalent, by def'n, to

vector space of multilinear fc'ns

$$\tau \in \underbrace{TM_p^* \times \cdots \times TM_p^*}_{r} \times \underbrace{TM_p \times \cdots \times TM_p}_{s}$$

$$\tau(\omega_1, \cdots, \omega_r, X_1, \cdots X_s) \in \mathbb{R}$$

covec. at p $\quad TM_p^*$ $\quad$ vec. at $\quad TM_p$

(let us p vary over M)

(r,s) - tensor field $T \in \Gamma(T_{r,s}M)$ is an $\mathbb{R}$-multilinear map

$$T : (\Gamma(TM^*))^r \times (\Gamma(TM))^s \longrightarrow C^\infty(M)$$

$$T(w_1, \dots, w_r, X_1, \dots X_s) \in C^\infty(M)$$

in addition is $C^\infty(M)$ homog. in each argument

$$T(w_1, w_2, fX_1, X_2, X_3) = f\,T(w_1, w_2, X_1, X_2, X_3)$$

$$\forall f \in C^\infty(M)$$

A $(1,s)$-tensor field T can be seen as a s-linear map $\tilde{T} : (\Gamma(TM)^s) \longrightarrow \Gamma(TM)$

$$T(w, X_1, X_2) \in C^\infty(M)$$

$$\tilde{T}(X_1, X_2) \in \Gamma(TM)$$

$$w(\tilde{T}(X_1, X_2)) := T(w, X_1, X_2) \quad \forall w$$

(Appendix C of 1rst set of notes)

$$T_1 \in \Gamma(T_{r_1, s_1} M) \ \& \ T_2 \in \Gamma(T_{r_2, s_2} M)$$

$$T_1 \otimes T_2 \in \Gamma(T_{r_1+r_2, s_1+s_2} M) \qquad \begin{aligned} r &= r_1 + r_2 \\ s &= s_1 + s_2 \end{aligned}$$

$$T_1 \otimes T_2 (\omega_1, \dots, \omega_r, X_1, \dots X_s) =$$

$$= T_1(\omega_1, \dots, \omega_{r_1}, X_1 \dots, X_{s_1}) T_2(\omega_{r_1+1}, \dots, \omega_r, X_{s_1+1}, \dots X_s)$$

In a chart $\quad \varphi : U \longrightarrow \varphi(U) \qquad T \in \Gamma(T_{r,s} M)$

$\quad U \subseteq M, \quad \varphi(U) \subseteq \mathbb{R}^m$

$$T|_U = \sum_{\substack{1 \leq i_1, \dots, i_r \leq m \\ 1 \leq j_1, \dots, j_s \leq m}} T^{i_1, \dots, i_r}_{j_1, \dots, j_s} \frac{\partial}{\partial \varphi^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial \varphi^{i_r}} \otimes d\varphi^{j_1} \otimes \cdots \otimes d\varphi^{j_s}$$

<u>Def'n 1.1</u> Riemann metric g on M is

a $(0,2)$- tensor field s.t

$$\forall p \in M \quad g_p : TM_p \times TM_p \longrightarrow \mathbb{R}$$

is an <u>inner product</u> (i.e <u>positive definite</u> <u>symmetric</u> ~~bilinear~~ form)

Given a chart $\varphi : U \to \varphi(U) \subset \mathbb{R}^m$ ($U \subset M$) the restriction

$$g|_U = \sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}} \underbrace{g_{ij}} \underbrace{d\varphi^i \otimes d\varphi^j}$$

$$(d\varphi^i \otimes d\varphi^j)(X,Y) = d\varphi^i(X)\, d\varphi^j(Y)$$
$$= (X\varphi^i)(Y\varphi^j)$$

X and Y
acting as derivations

$$g_{ij} = g\left(\frac{\partial}{\partial \varphi^i}, \frac{\partial}{\partial \varphi^j}\right) \in C^\infty(U) \quad \text{for all fixed } i,j$$

As a matrix (g_{ij}) is $>$ definite & symmetric.

<u>Riem. manifold</u> is a pair (M,g)

- M smooth mfld
- g Riem. metric on M

Rem 1.2 On every smooth mfld M $\exists$ "many" Riem. metric g (exercise: use local coord + partition of unity)

$(\bar{M}, \bar{g})$ Riem. mfld $F: \underset{\text{smooth mfld}}{M} \longrightarrow \bar{M}$ is an immersion

pull-back metric $F^*\bar{g}$ on M

$$(F^*\bar{g})_p(\underbrace{v, w}_{TM_p}) := \bar{g}_{F(p)}(F_*v, F_*w)$$

$$= \bar{g}_{F(p)}(dF_p(v), dF_p(w))$$

is a Riemann metric on M

Def'n Two Riem. mfld (M, g) and $(\bar{M}, \bar{g})$ are isometric iff $\exists$ diffeo $F : M \longrightarrow \bar{M}$ st.

$$F^* \bar{g} = g \quad \left(\Longleftrightarrow \quad (F^{-1})^* g = \bar{g} \right)$$

Such F is called isometry

Length $\quad c : [a, b] \longrightarrow M$, $s \in [a,b]$

$$L(c) = \int_a^b \sqrt{g(\dot{c}, \dot{c})}\, ds$$

$$\boxed{\dot{c} \in \Gamma(c^* TM)}$$

$$\dot{c}(s) = dc_s(1)$$

$$\dot{c}(s) \in TM_{c(s)}$$

Exercise

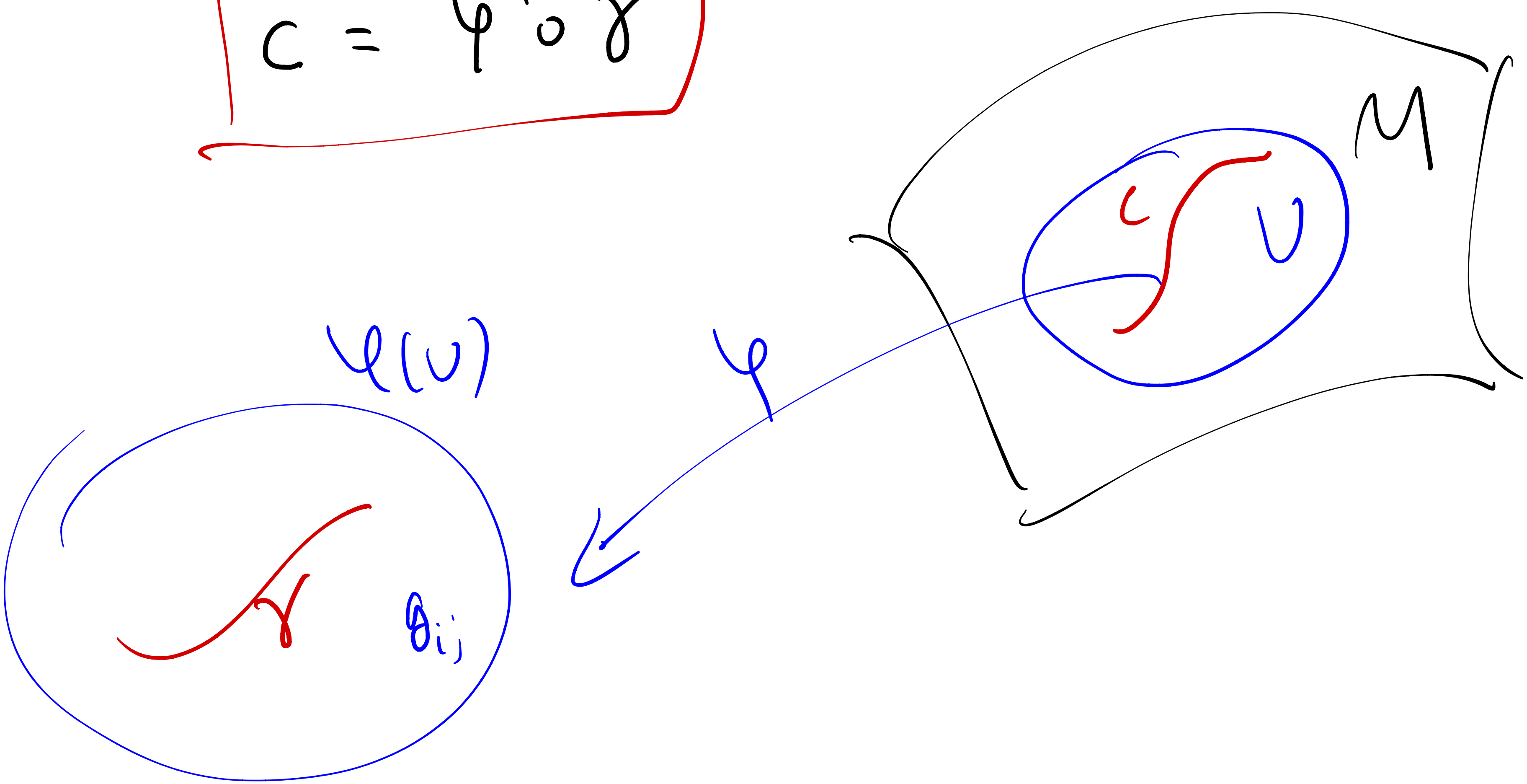
- this is independent of reparam.
- $c([a,b]) \subset U$

  & $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ chart

$\tilde{c}(t) = c \circ s(t)$

($s : [c,d] \to [a,b]$ bijective)

$$\boxed{c = \varphi^{-1} \circ \gamma}$$

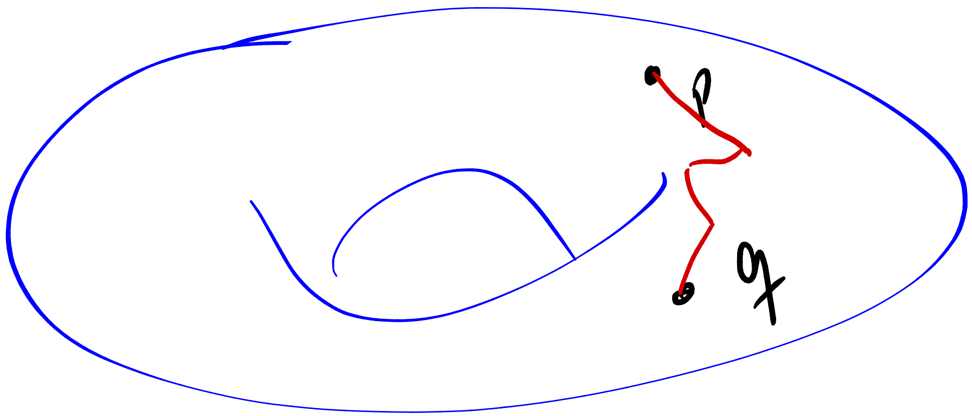


$$g|_U = \sum_{i,j} g_{ij}\, d\varphi^i \otimes d\varphi^j$$

$$L(c) = \int_a^b \sqrt{\sum_{i,j} (g_{ij} \circ \gamma)\, \dot\gamma^i \dot\gamma^j}\, ds$$

<u>Metric structure</u> $p, q \in M$

$$d(p,q) = \inf \left\{ L(c) : \begin{array}{l} c : [0,1] \longrightarrow M \\ \text{piecewise smooth curve} \\ (\text{\& continuous})\ c(0) = p \text{ and } c(1) = q \end{array} \right\}$$



Thm 1.4 (dist fcn)

d is a distance on every connected Riem. mfld.

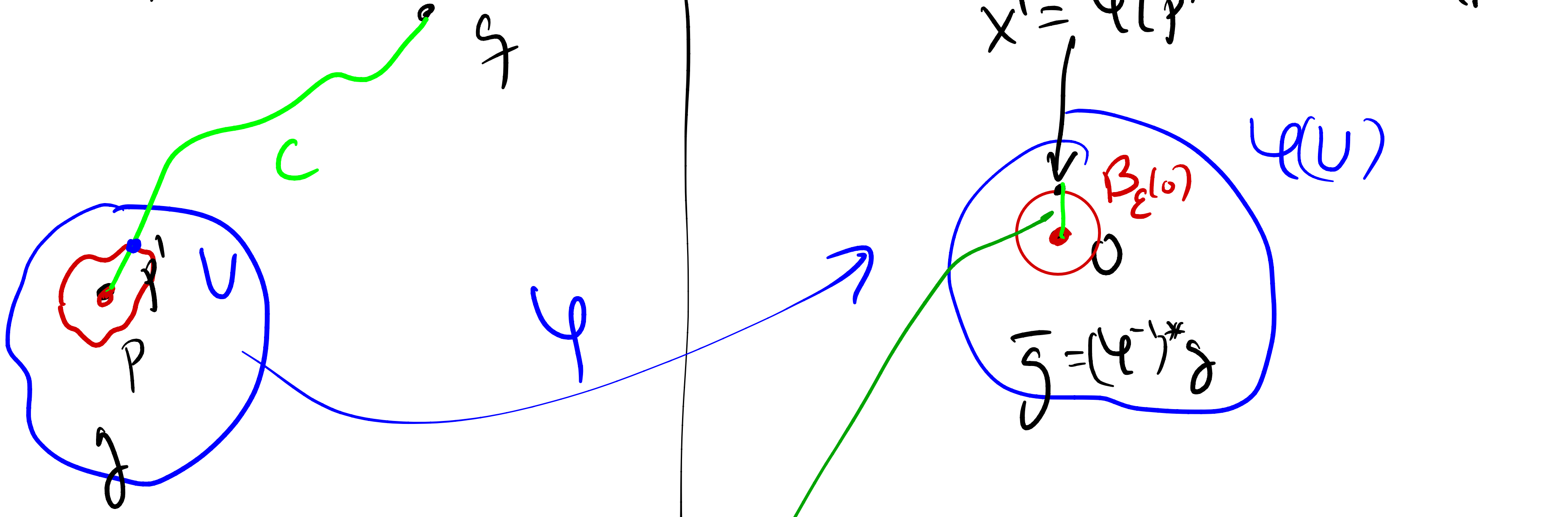
1) $d(p,q) < \infty$ (connectedness!)

2) $d(p,p') + d(p',q) \geq d(p,q)$

3) $d(p,p) = 0$ & $d(p,q) = d(q,p)$

4) $d(p,q) > 0$ for $p \neq q$

proof of 4)



$\varphi(p) = 0$

$c : [0,1] \to M$

$c(\tau) = p'$

$$\gamma = \varphi \circ c|_{[0,\tau]}$$

g is $>$ definite [x in cpt subset of $\varphi(U)$]

$$\bar{g}_x(\xi,\xi) \geq \lambda^2 \langle \xi,\xi \rangle_{\mathbb{R}^m}$$

for some $\lambda^2 > 0$.

$$L(c) \geq L(c|_{[0,\tau]}) = \int_0^\tau \sqrt{\bar{g}(\dot{\gamma},\dot{\gamma})}\, ds$$

$$\geq \lambda \int_0^\tau |\dot{\gamma}|_{\text{Eucl.}}\, ds \geq \underbrace{\lambda \varepsilon}_{} > 0$$

same number for all c

$\blacksquare$

A crucial difference between $\mathbb{R}^n$ and M, is that in the latter, we <u>cannot</u>:

- subtract points $\quad p, q \in M \longrightarrow p - q = ??$
- subtract tangent vect at different point $\quad v \in TM_p,\ w \in TM_q \quad v - w = ??$

$$c : [0, T] \longrightarrow M \qquad c'(t) \in TM_{c(t)}$$

$$c''(t) \overset{?}{=} \lim_{s \downarrow 0} \frac{c'(t+s) - c'(t)}{s}$$

For $\mathbb{R}^n$ Euclidean coordinates

$\phi^t(x) = \phi(t,x)$

$$(D_X Y)_p = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i} Y_p$$

ϕ^t vector flow of X

$$\boxed{\frac{d}{dt}\phi = X \circ \phi}$$

$$= \frac{d}{dt}\Big|_{t=0} Y_{\phi^t(p)}$$

$$= \lim_{t \downarrow 0} \frac{Y_{\phi^t(p)} - Y_p}{t}$$

Natural "generalization" (1st try)

$$(\mathcal{L}_X Y)_p = \lim_{t \downarrow 0} \frac{1}{t} \left((d\phi^{-t})_{\phi^t(p)} (Y_{\phi^t(p)}) - Y_p \right) \in TM_p$$

Lie bracket

$$M \supset U$$ open nbhd of p

$X, Y \in \Gamma(TM)$, $p \in M$, $f \in C^\infty(U_p)$

$$[X,Y]_p(f) := X_p(Y(f)) - Y_p(X(f))$$

defines a derivation at p:

- $Z = [X,Y]_p$ is $\mathbb{R}$-linear $\quad (Z(af+bg) = aZf + bZg)$
- $Z(f \cdot g) = Zf\, g(p) + f(p)\, Zg \quad$ (exercise)

$$\Rightarrow \quad [X,Y]_p \in TM_p$$

$[X,Y] \in \Gamma(TM)$ (we will see it is smooth in a moment)

Thm 10.11 (exercise)

(1) $[\cdot,\cdot]$ is $\mathbb{R}$-bilinear

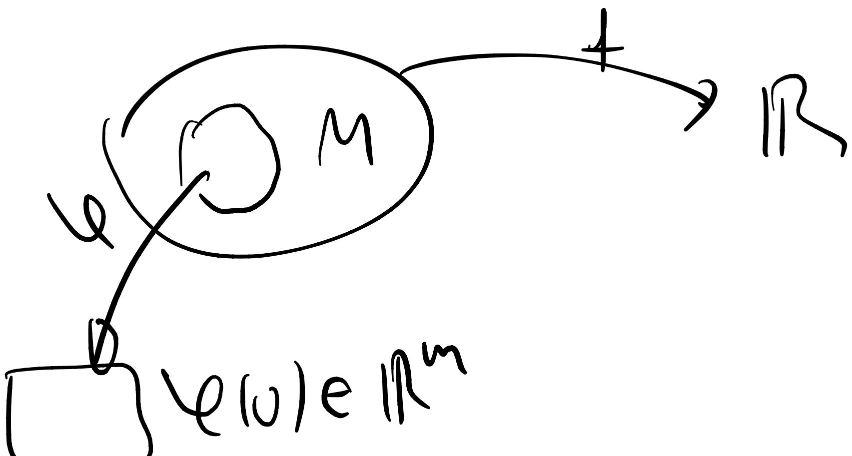
(2) $[fX, gY] = fg[X,Y] + fX(g)Y - gY(f)X$

(3) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

In a chart (φ, U) (of M)

$f \in C^\infty(M)$, $g = f|_U \circ \varphi^{-1}$

$$\left(\frac{\partial}{\partial \varphi^i} f\right) \circ \varphi^{-1} = \frac{\partial}{\partial x^i} g$$



Observation

$$\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}\right) g \equiv 0$$

$$\Rightarrow \left[\frac{\partial}{\partial \varphi^i}, \frac{\partial}{\partial \varphi^j}\right] \equiv 0$$

$$X|_U = \sum_{i=1}^{n} X^i \frac{\partial}{\partial \varphi^i} \qquad Y|_U = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial \varphi^j}$$

$$[X,Y] = \sum_i \left(\sum_j X^j \frac{\partial Y^i}{\partial \varphi^j} - Y^j \frac{\partial X^i}{\partial \varphi^j} \right) \frac{\partial}{\partial \varphi^i}$$

Thm 10.12 (Lie derivative) If ϕ^t is the local flow of X around p then

$$[X,Y]_p = \lim_{t \to 0} \frac{1}{t} \left(d\phi^{-t} \left(Y_{\phi^t(p)} \right) - Y_p \right)$$

$$= \frac{d}{dt}\Big|_{t=0} d\phi^{-t} \left(Y_{\phi^t(p)} \right)$$

$$= (\mathcal{L}_X Y)_p$$

Vector flows on M $\quad X \in \Gamma(TM)$

$c : (a,b) \longrightarrow M$ integral curve of X if

$$\dot c(t) = X_{c(t)} \quad (*)$$

$$\boxed{\xi^i := X^i \circ \varphi^{-1}}$$

In a chart (φ, U) $\quad X|_U(p) = \sum X^i(p) \frac{\partial}{\partial \varphi^i}\Big|_p$

$(*) \iff \boxed{\dot\gamma(t) = \xi_{\gamma(t)}}$ for $\gamma = \varphi \circ c$

Thm 10.8 (local flow) $\forall p \in M$ $\exists U$ open nbh. and $\varepsilon > 0$ s.t. $\forall q \in U$ there is a unique integral curve $c_q : (-\varepsilon, \varepsilon) \to M$ of X with $c_q(0) = q$.

The map $\phi : (-\varepsilon, \varepsilon) \times U \to M$

$$\phi(t, q) = \phi^t(q) := c_q(t)$$ is <u>smooth</u>

<u>proof</u> ODE theory you know $\blacksquare$

Recall that uniqueness

$$\phi^t(\phi^s(q)) = \phi^{s+t}(q)$$

whenever $s, t, s+t \in (-\varepsilon, \varepsilon)$, $q \in U$, $\phi^s(q) \in U$

In particular $V \subset U$ open nh. of q and, $|s|$ small so that

$$\phi^s : V \to \phi^s(V) \subset U \quad (C^\infty \text{ diffeo})$$

$$\phi^{-s} \circ \phi^{s}\big|_V = \phi^{0}\big|_V = \mathrm{id}_V$$

A v.f. is <u>completely integrable</u> if $\forall f \in M$ $\exists$ integral curve $c_f : \mathbb{R} \longrightarrow M$ of X with $c_f(0) = f$

If this happens $\phi : \mathbb{R} \times M \longrightarrow M$ is a 1-param. family of diffeomorphisms.

<u>Thm. 10.9</u> Every $X \in \Gamma(TM)$ with cpt support is completely integrable.

proof $\forall p \in M$ $\exists$ nbhd U_p and $(-\varepsilon_p, \varepsilon_p)$ as given by Thm 10.8

$$\mathrm{spt}(X) \text{ cpt} \Rightarrow \mathrm{spt}(X) \subset \bigcup_{i=1}^{K} U_{p_i}$$

$$\varepsilon := \min\{\varepsilon_{p_i}, \ 1 \leq i \leq K\}$$

$$\underset{10.4}{\Rightarrow} \quad \phi^t(q) \text{ is defined on } (-\varepsilon, \varepsilon) \times M$$

$$[X_q = 0 \Rightarrow \phi^t(q) \equiv q \quad \forall t \in \mathbb{R}]$$

thus, we can define for $t \in \mathbb{R}$

$$t = j\frac{\varepsilon}{2} + r \qquad j \in \mathbb{Z}$$

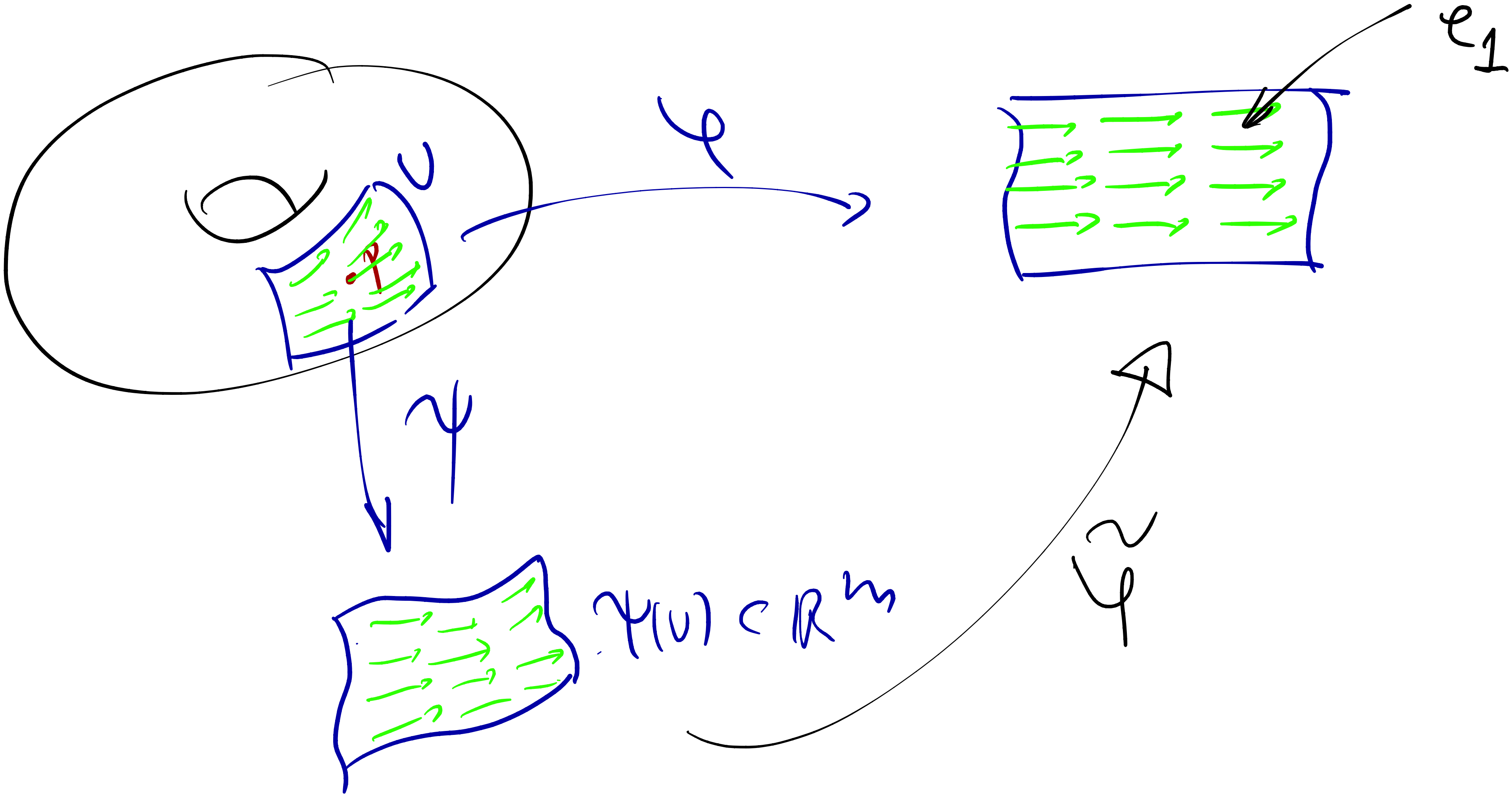
Put $\phi^t := \phi^r \circ (\phi^{\varepsilon/2})^j$ $\blacksquare$

Last preliminary before pf. Thm 10.12

<u>Lemma 10.10</u> (flow-box) $X \in \Gamma(TM)$, $p \in M$,

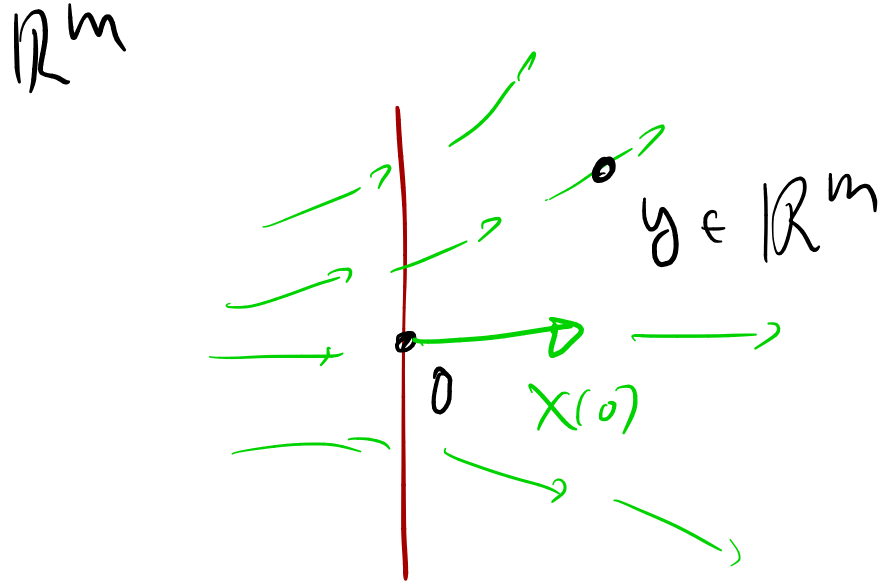
$X_p \neq 0 \implies \exists$ chart (φ, U) around p

s.t $X|_U = \frac{\partial}{\partial \varphi^1}$



Assume w.l.o.g. X v.f. in open nbhd of $0 \in \mathbb{R}^n$

$X(0) = e_1$

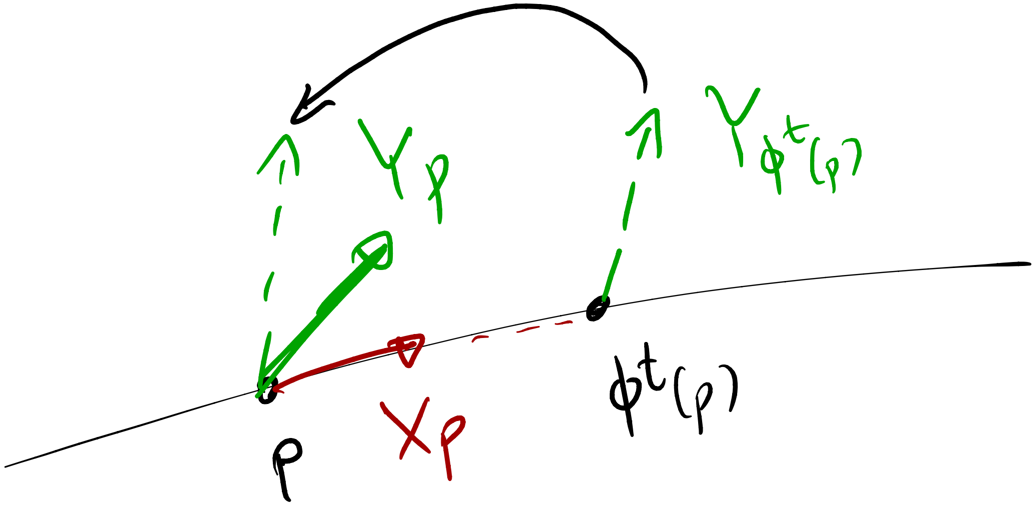


$$\psi(y) = (t, x^2, \dots, x^m)$$

$$y = \phi^t(0, x^2, \dots, x^m)$$

ψ satisfies what we want $\blacksquare$

<u>proof of Thm 10.12</u>



<u>Case 1</u> $X_p \neq 0$: use flow box (U, φ)

$$X|_U = \frac{\partial}{\partial \varphi^1}$$

$$Y|_U = \sum_{i=1}^{m} Y^i \frac{\partial}{\partial \varphi^i}$$

$$[X,Y]_p = \sum_{i=1}^{m} \underbrace{\frac{\partial Y^i}{\partial \varphi^1}(p)} \frac{\partial}{\partial \varphi^i}\Big|_p$$

$$= \sum_{i=1}^{m} \lim_{t \to 0} \frac{1}{t} \left(Y^i(\phi^t(p)) - Y^i(p) \right) \frac{\partial}{\partial \varphi^i}\Big|_p$$

$$= \lim_{t \to 0} \frac{1}{t} \left(\sum_i Y^i(\phi^t(p)) \underbrace{\frac{\partial}{\partial \varphi^i}\Big|_p} - Y(p) \right)$$

$\varphi \circ \phi^t$ is <u>translation</u> $\longrightarrow$ $= d(\phi^{-t})\left(\frac{\partial}{\partial \varphi^i}\Big|_{\phi^t(p)} \right)$

$$= (\mathcal{L}_X Y)_p$$

<u>Case 2</u> $X_p = 0$, given $f \in C^\infty(M)$

$$[X,Y]_p(f) \underset{(*)}{=} -Y_p(Xf) = -\frac{d}{ds}\Big|_{s=0} (Xf)(c(s))$$

where $s \mapsto c(s)$ is an int. curve of Y $\quad c(0) = p$

ϕ^t local flow of X

$$\underset{(*)}{=} \frac{d}{ds}\Big|_{s=0} \left(-X_{c(s)}(f)\right)$$

$$= \frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} (f \circ \phi^{-t})(c(s))$$

$$= \frac{d}{dt}\Big|_{t=0} Y_p\left(f \circ \phi^{-t}\right)$$

$$= \frac{d}{dt}\Big|_{t=0} \Big(d(\phi^{-t})(Y_p)\, f \Big) = (\alpha_X Y)_p f$$

$p \equiv \phi^t(p)$

$\blacksquare$

The tangent space revisited

$p \in M^m \subseteq \mathbb{R}^n$ smooth submfld

$v \in TM_p \subset \mathbb{R}^n \iff \exists q_k \in M, \exists r_k \downarrow 0$ s.t. $\dfrac{q_k - p}{r_k} \longrightarrow v$

$\iff \forall f \in C_c^\infty(\mathbb{R}^n)$

$\dfrac{f(q_k) - f(p)}{r_k} \longrightarrow df_p(v)$

On "abstract" mfld M (smooth)

$$TM_p := \left\{ \{(q_k, r_k)\}_{k \geq 1} : \forall f \in C_c^\infty(M), \frac{f(q_k) - f(p)}{r_k} \text{ is convergent} \right\} \Big/ \sim_p$$

$(q_k, r_k) \sim_p (q'_k, r'_k)$ iff $\forall f$

$$\lim \frac{f(q_k) - f(p)}{r_k} = \lim \frac{f(q'_k) - f(p)}{r'_k}$$

Covariant der. for submflds $M^m \subset \mathbb{R}^n$

$Y = (Y^1, \dots Y^n)$

acting as derivation

$X, Y \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$

$$D_X Y = (X Y^1, \dots, X Y^n)$$

$c : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$

$c(0) = p$

$c'(0) = X(p)$

$$\frac{D}{dt}\Big|_{t=0} Y = D_{X(p)} Y$$

$$= \lim_{t \downarrow 0} \frac{Y(c(t)) - Y(p)}{t}$$

$$X = (X^1, \dots X^n) \qquad X Y^i = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j} Y^i$$

$X, Y \in \Gamma(TM)$ <u>extend</u> then $\tilde{X}, \tilde{Y} \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$

$$D_X Y(p) = \left(D_{\tilde{X}} \tilde{Y}(p) \right)^T \leftarrow$$

orthogonal projection onto

$\mathbb{R}^n \to TM_p$

<u>Def'n</u> 1.5 M m-dim mfld. A <u>connection</u> ∇ on TM is a $\mathbb{R}$-bilinear map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$

[Notation: $\nabla_X Y$ instead of $\nabla(X,Y)$]

(1) $\nabla_{fX} Y = f \nabla_X Y$

(2) $\nabla_X (fY) = (Xf) Y + f \nabla_X Y$

$\left(\forall X, Y \in \Gamma(TM),\ \forall f \in C^\infty(M) \right)$

<u>exercise</u> $(\nabla_X Y)_p$ depends only on X_p and $Y|_U$

where U is any smooth open nbhd of p.

If $A_1, \dots, A_m \in \Gamma(TU)$ $\quad U \subset M$ open set

s.t $A_1, \dots A_m$ at p are a basis of TM_p

[e.g. (U, φ) chart $\quad A_i = \frac{\partial}{\partial x^i} \quad i=1, \dots, m$]

$$\nabla_{A_i} A_j =: \sum_{k=1}^{m} \underbrace{\Gamma_{ij}^{k}} A_k \qquad \text{Christoffel symbols}$$

<u>Lemma 1.6</u> $\quad X|_U = \sum_{i=1}^{m} X^i A_i \quad , \quad Y|_U = \sum_{j=1}^{m} Y^j A_j$

$$\nabla_X Y = \sum_{k=1}^{m} \left[X(Y^k) + \sum_{i,j} X^i Y^j \Gamma_{ij}^{k} \right] A_k$$

Remark In order to compute $\nabla_X Y(p)$

we only need X_p and $Y|_{\text{image}(c)}$

$c : (-\varepsilon, \varepsilon) \to M$ is any curve with $c(0) = p$, $c'(0) = X_p$

Def'n 1.8 M C^∞ mfld, ∇ connection on TM

(1) The map $T : (\Gamma(TM) \times \Gamma(TM)) \to \Gamma(TM)$

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$$

is called torsion of ∇. If $T \equiv 0$, ∇ is torsion free

(2) Riem. metric $g = \langle \cdot , \cdot \rangle$ on M, D is <u>compatible</u> (with g) if $\forall X, Y, Z \in \Gamma(TM)$

$$Z\langle X, Y\rangle = \langle D_Z X, Y\rangle + \langle X, D_Z Y\rangle$$

<u>Exercise</u> If $f \in C^\infty(M)$

$$T(X,Y) = -T(Y,X)$$

$$T(fX, Y) = f\,T(X,Y)$$

so T is a antisymmetric <u>tensor</u>

Thm-defn 1.9 (Levy-Civita connection)

For every (M,g) Riem. mfld $\exists$ a unique conection on TM that is torsion free & compatible.

Moreover, this conn. is characterized by "Koszul's formula"

$$2\langle \underbrace{D_X Y}, Z\rangle := \underbrace{X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle} - \langle X,[Y,Z]\rangle - \langle Y,[X,Z]\rangle + \langle Z,[X,Y]\rangle \qquad (*)$$

This connection is the Levy-Civita conn. (the notation is then $D_X Y$ instead of $\nabla_X Y$)

proof <u>Step 1</u> Let us show that is a connection is compatible and torsion free then it must satisfy (*)

compatibility

$$X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle =$$

$$\langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle + \langle D_Y X, Z\rangle + \langle X, D_Y Z\rangle - \langle D_Z X, Y\rangle - \langle X, D_Z Y\rangle$$

$\langle\, ,\rangle$ symmetric bilinear

$$= \langle X, D_Y Z - D_Z Y\rangle + \langle Y, D_X Z - D_Z X\rangle + \langle Z, D_X Y + D_Y X\rangle$$

D torsion free

$$= \langle X, [Y,Z] \rangle + \langle Y, [X,Z] \rangle + 2\langle Z, D_X Y \rangle - \langle Z, [X,Y] \rangle$$

<u>Step 2</u> Take (*) as def'n of $D_X Y$ and check that it is a connection, compatible, torsion free.

For example, let us check $\forall Z \in \Gamma(TM)$

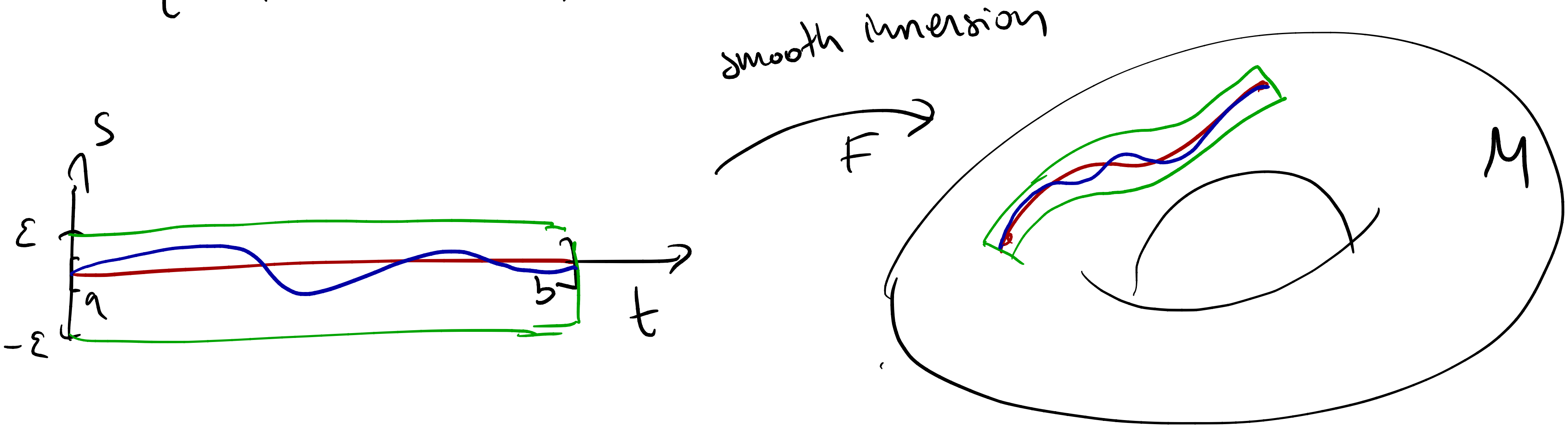
$$D_{fX} Y = f D_X Y \iff \langle D_{fX} Y, Z \rangle = \langle f D_X Y, Z \rangle$$

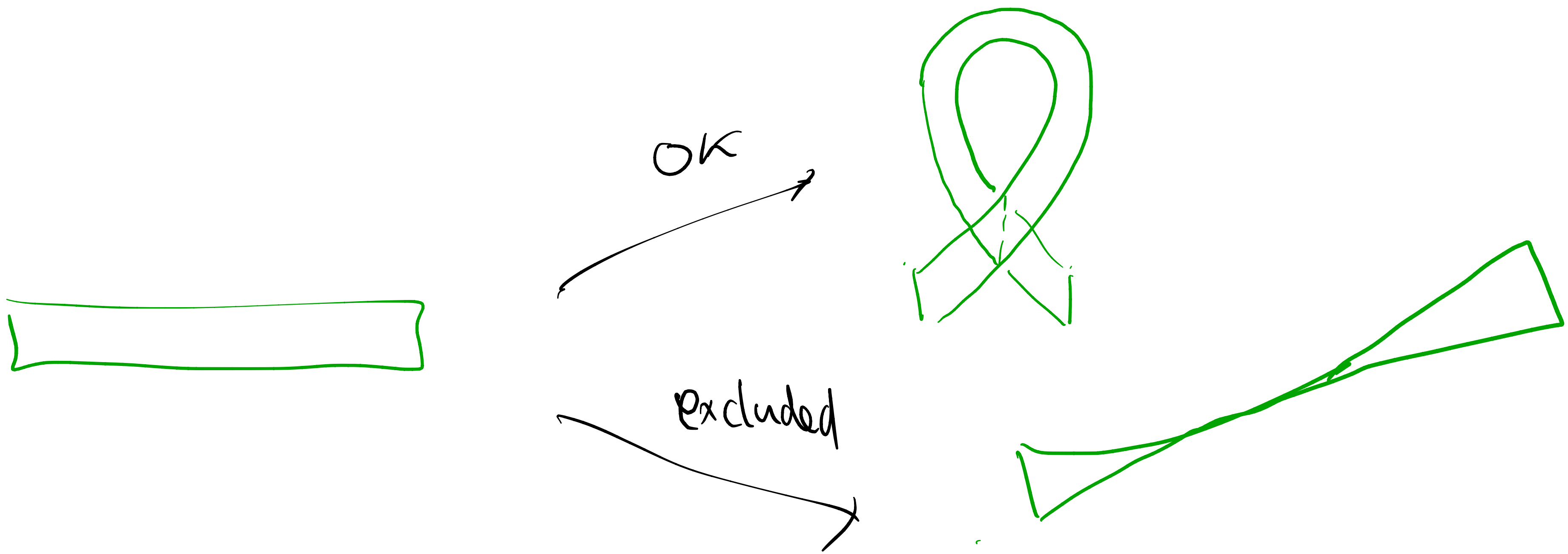
$$
\begin{aligned}
2\langle D_{fX}Y, Z\rangle = {} & fX\langle Y, Z\rangle + \overbrace{Y\langle fX, Z\rangle}^{Y(f\langle X, Z\rangle)} - \overbrace{Z\langle fX, Y\rangle}^{Z(f\langle X, Y\rangle)} \\
& - \langle fX, [Y, Z]\rangle - \langle Y, \underbrace{[fX, Z]}_{f[X,Z]-(Zf)X}\rangle + \langle Z, [\underbrace{fX, Y}_{f[X,Y]-(Yf)X}]\rangle
\end{aligned}
$$

$$
\begin{aligned}
= {} & f\big(X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle\big) \\
& + (Yf)\langle X, Z\rangle - Z f\langle X, Y\rangle \\
& + f\Big(-\langle X, [Y, Z]\rangle - \langle Y, [X, Z]\rangle + \langle Z, [X, Y]\rangle\Big) \\
& + \langle Y, (Zf)X\rangle - \langle Z, (Yf)X\rangle \qquad \blacksquare
\end{aligned}
$$

$$\begin{aligned} c : [a,b] &\longrightarrow M \\ t &\longmapsto c(t) \end{aligned}$$

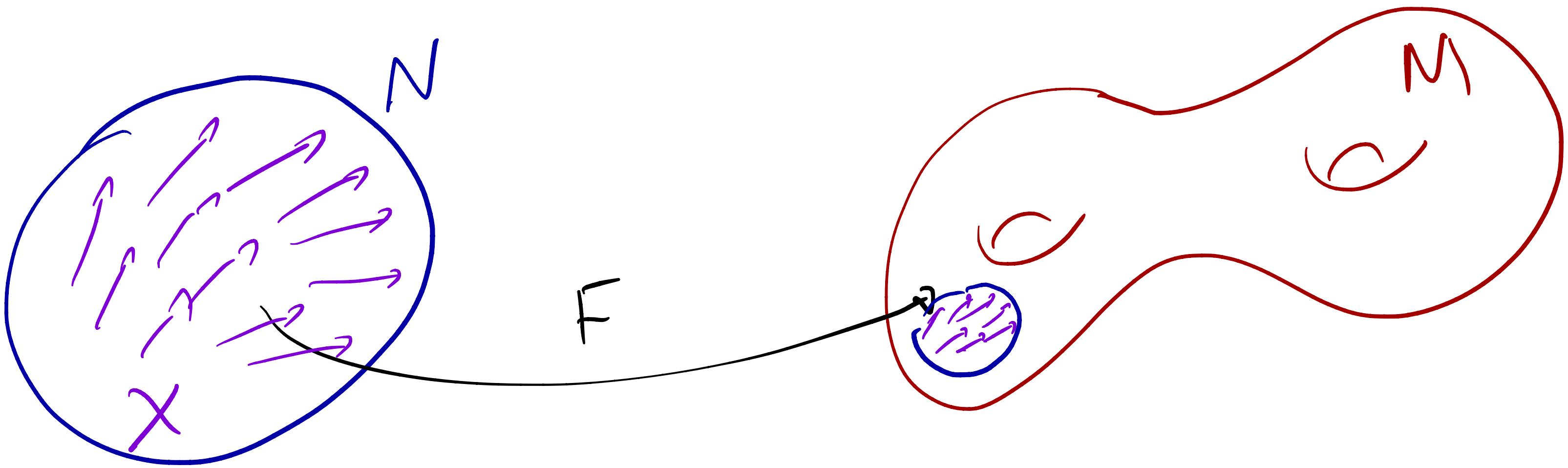
$c' \neq 0$ <u>regular</u> curve





Pushforward of vector fields

Given $F: N^n \to M^m$ smooth map (N, M smooth mflds)



In general, not possible to identify the push-forward of v.f $X \in \Gamma(TN)$ with some $F_* X \in \Gamma(TM)$

- F not surjective ($F_* X$ only defined on submanifold)
- F is not injective ($F_* X$ would be multiply defined)

How we "solve" these issues?

<u>Def'n</u> $Y \in \Gamma(F^*TM)$ $\left[\begin{array}{l} \text{i.e. } Y : N \to TM \text{ smooth} \\ Y(p) \in TM_{F(p)} \end{array}\right]$

is the push forward of $X \in \Gamma(TM)$

if $\quad Y(F(p)) = dF_p(X_p)$

we denote $Y = F_* X$

<u>pull-back section</u> $\quad Z \in \Gamma(TM) \qquad F^* Z = Z \circ F \in \Gamma(F^*TM)$

<u>Z, X are F related</u> if $\quad F^* Z = F_* X$

(in other words $dF_p(X) = Z_{F(p)}$)

if F is diffeomorphism each v.f $X \in \Gamma(TN)$ has a (unique) $Z \in \Gamma(TM)$ that is F related to X.

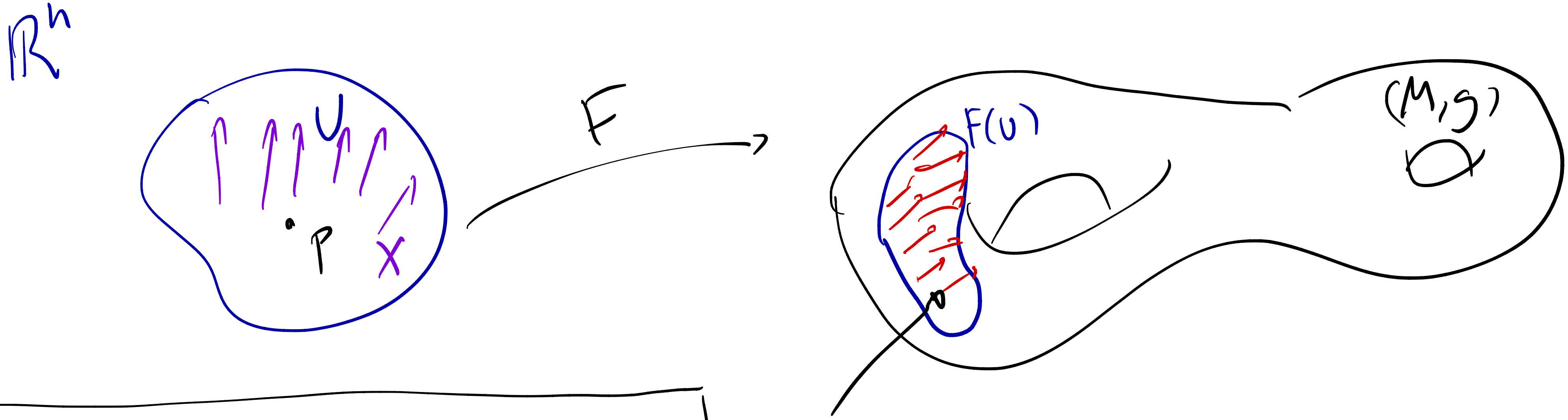
(M, g) Riemannian mfld, F immersion, N manifold

$$F : N \longrightarrow M$$

$$X \in \Gamma(TN), \quad Z \in \Gamma(F^* TM), \quad p \in N$$

$$(D_X Z)(p) := (D_{\tilde{X}} \tilde{Z})(F(p))$$

- U open nbhd of p (in N) s.t $F|_U$ is embedding.
- $\tilde{X}$ smooth extension of $(F_* X) \circ F^{-1} \big|_{F(U)}$
- $\tilde{Z}$ " " of $Z \circ F^{-1} \big|_{F(U)}$



In particular: $X, Y \in \Gamma(TN)$

$z \circ F^{-1}|_{F(U)}$

$$D_X(F_* Y) = (D_{\tilde{X}} \tilde{Y}) \circ F^{-1}|_{F(U)}$$

$\tilde{X}, \tilde{Y}$ are extensions $(F_* X) \circ F^{-1}|_{F(U)}$, $(F_* Y) \circ F^{-1}|_{F(U)}$

If $N = U$ open set in $\mathbb{R}^n$

$$X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$$

$$\frac{D}{\partial x^i} := D_{\frac{\partial}{\partial x^i}} \implies D_X = \sum X^i \frac{D}{\partial x^i}$$

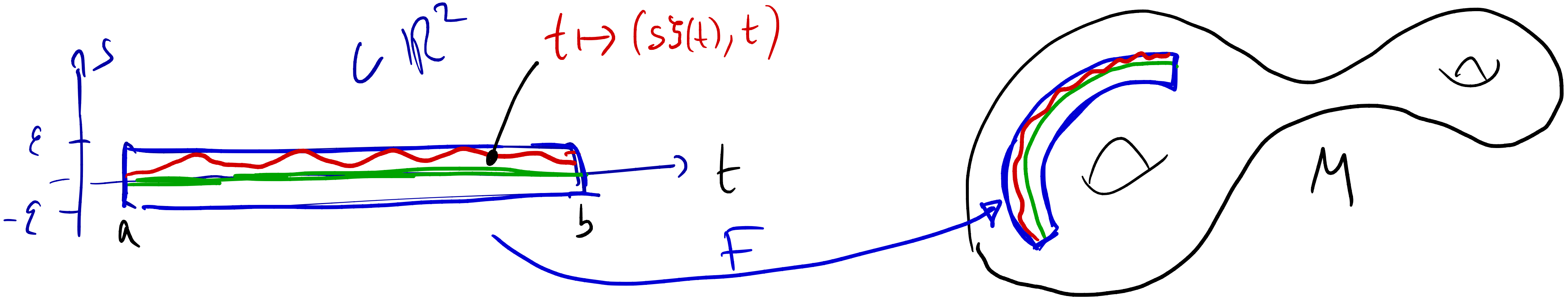
Exercise $F : N \longrightarrow (M, g)$

$$Z_1, Z_2 \in \Gamma(F^* TM), \quad X, Y \in \Gamma(TN)$$

(1) $$X\langle Z_1, Z_2\rangle = \langle D_X Z_1, Z_2\rangle + \langle Z_1, D_X Z_2\rangle$$

(2) $$D_X(F_* Y) - D_Y(F_* X) - [X, Y] \equiv 0$$

(hint use $[F_* X, F_* Y] = F_*[X, Y]$)



$$\boxed{c(t) = F(0,t)}$$

$$\gamma_s(t) = F(s\xi(t), t)$$

$\xi : [a,b] \longrightarrow (0,1)$ a given smooth fc'n

<u>Goal</u> compute $\left.\dfrac{d}{ds}\right|_{s=0} L(\gamma_s)$ $\quad (\gamma_s(\cdot) = \gamma(s,\cdot))$

We could here put simply $\gamma_s(t) = F(s,t)$ (up to changing F)

<u>Thm 1.15</u> $|\dot{c}(t)| = \lambda, \quad \forall t \in [a,b]$ $\qquad \left[\dot{} = \frac{\partial}{\partial t}, \frac{d}{dt} \right]$

$$\frac{d}{ds}\Big|_{s=0} L(\gamma_s) = \frac{1}{\lambda}\left[\langle V_0(t), \dot{c}(t) \rangle \Big|_a^b - \int_a^b \langle V_0(t), \frac{D}{dt}\dot{c}(t) \rangle dt \right]$$

where $V = F_*(\xi \frac{\partial}{\partial s})$, $\quad V_s = V(s, \cdot)$.

<u>proof</u> $\qquad \left[\dot{\gamma}_s(t) = (F_* T)(s\xi(t), t) \right]$ $\qquad |\dot{\gamma}_s(t)| = \sqrt{\langle \dot{\gamma}_s(t), \dot{\gamma}_s(t) \rangle}$

$$\frac{\partial}{\partial s} |\dot{\gamma}_s(t)| = \frac{1}{2|\dot{\gamma}_s(t)|} \frac{\partial}{\partial s} \langle \dot{\gamma}_s(t), \dot{\gamma}_s(t) \rangle$$

$$= \frac{1}{|\dot{\gamma}_s(t)|} \langle \xi \frac{D}{\partial s} F_* T, \dot{\gamma}_s(t) \rangle$$

$$\frac{d}{ds}\Big|_{s=0} L(\gamma_s) = \frac{d}{ds}\Big|_{s=0} \int_a^b |\dot\gamma_s(t)|\, dt$$

$$\boxed{D_V F_* T = D_T F_* V}$$

$|\dot\gamma_0(t)| = \lambda$

$$= \frac{1}{\lambda}\int_a^b \Big\langle \underbrace{\xi \frac{D}{\partial s}}_{V} F_* T \,,\ \dot\gamma_s(t)\Big\rangle\Big|_{s=0} dt$$

D is torsion-free

$$= \frac{1}{\lambda}\int_a^b \Big\langle \frac{D}{\partial t} F_*\Big(\xi(t)\frac{\partial}{\partial s}\Big),\ \dot\gamma_s(t)\Big\rangle\Big|_{s=0} dt$$

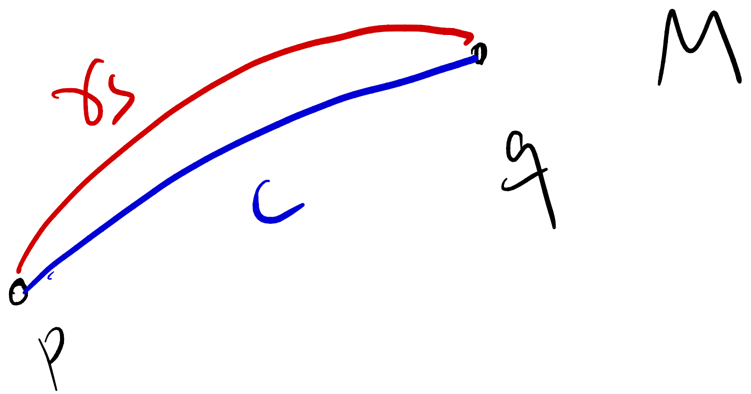
$$= \frac{1}{\lambda}\int_a^b \frac{d}{dt}\Big\langle \overbrace{\xi(t)\, F_*\frac{\partial}{\partial s}\Big|_{s=0}}^{V_0(t)},\ \dot c(t)\Big\rangle$$

$$- \frac{1}{\lambda}\int_a^b \Big\langle \underbrace{\xi(t)\, F_*\frac{\partial}{\partial s}\Big|_{s=0}}_{V_0(t)},\ \frac{D}{dt}\dot c(t)\Big\rangle\, dt$$

$$= \frac{1}{\lambda}\left[\langle V_0(t), \dot{c}(t) \rangle \right]_a^b - \frac{1}{\lambda}\int_a^b \left\langle V_0, \frac{D}{dt}\dot{c}(t) \right\rangle dt$$

∎

<u>Remark</u> As in DG I, if $c : t \longmapsto c(t)$ is a smooth reg. curve (const speed $|\dot{c}| = \lambda$) that minimizes length



The for any <u>proper variation</u> "limit case" $\xi(a) = \xi(b) = 0$

we have

$$0 = \frac{d}{ds}\Big|_{s=0} L(\gamma_s) = -\frac{1}{\lambda}\int_a^b \left\langle V_0, \frac{D}{dt}\dot{c}(t)\right\rangle dt$$

Since $\frac{D}{dt}\dot{c}(t)$ is smoot (in part C^0)

$$\Rightarrow \quad \frac{D}{dt}\dot{c}(t) \equiv 0 \quad \Leftrightarrow \quad \text{geodesic eq'n}$$

$\Leftrightarrow$ $\dot{c}(t)$ is <u>parallel</u> (along c)

$Y \in \Gamma(c^* TM)$ is <u>parallel</u> if $\frac{D}{dt} Y \equiv 0$

In local coordinates (φ, U) $\left[\text{suppose } c : I \to U\right]$

$$\varphi \circ c(t) = \left(x^1(t), \dots, x^m(t)\right) \implies (\varphi \circ c)' = \sum_{i=1}^m \dot{x}^i e_i$$

canonic basis of $\mathbb{R}^n$

$$Y(t) = \sum_j Y^j(t) \left(\frac{\partial}{\partial \varphi^j} \circ c\right)$$

$$\frac{D}{dt} Y = \sum_{k=1}^m \left(\dot{Y}^k + \sum_{i,j=1}^m \dot{x}^i Y^j \left(\Gamma_{ij}^k \circ c\right)\right) \frac{\partial}{\partial \varphi^k} \circ c$$

In particular $c(t)$ is a geodesic, φ chart, $\boxed{x = \varphi \circ c}$

$$Y(t) = \dot{c}(t) \Rightarrow Y^k = \dot{x}^k$$

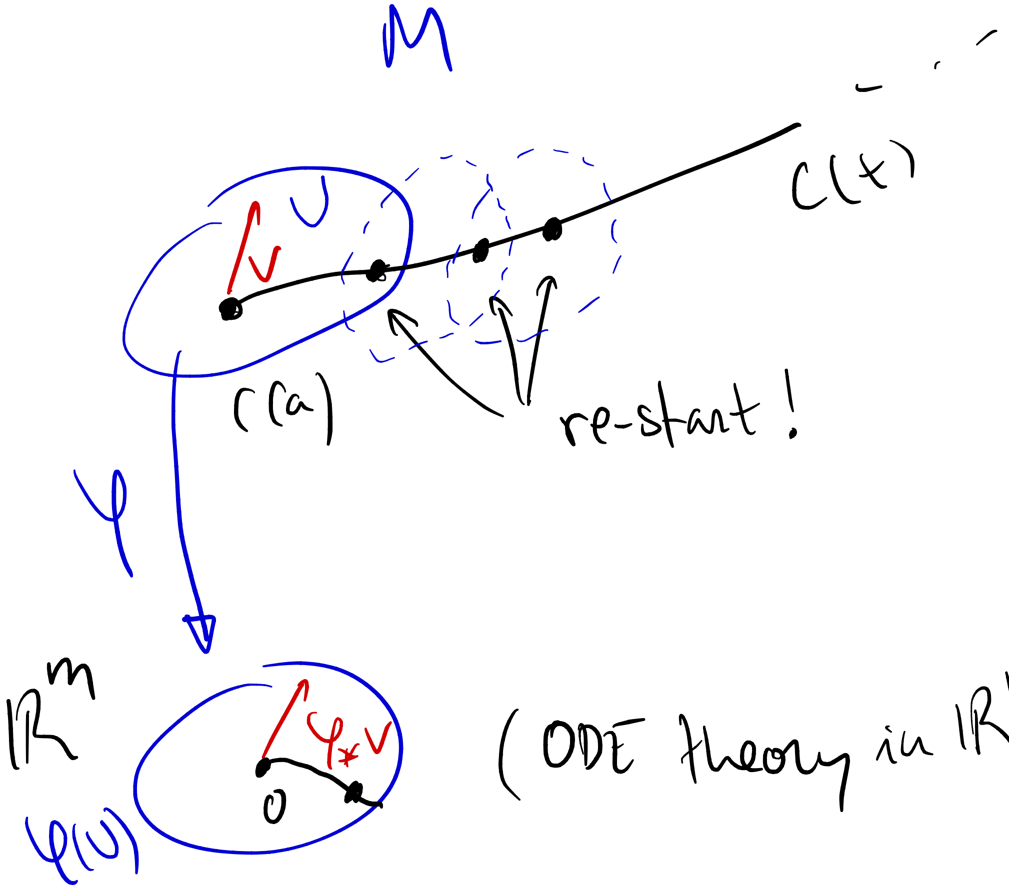
$$\frac{D}{dt}\dot{c} \equiv 0 \iff \left(\ddot{x}^k + \sum_{i,j=1}^{n} \dot{x}^i \dot{x}^j \left(\Gamma^k_{ij} \circ c\right)\right) = 0$$

<u>geodesic eq'n in coordinates</u>

Standard ODE theory implies:

(1) <u>Prop 1.13</u> Given $c : [a,b] \to M$ a C^1 curve, for every $v \in TM_{c(a)}$ $\exists!$ parallel v.f. Y^c_v along c with $Y^c_v(a) = v$.

proof



($\text{ODE theory in } \mathbb{R}^m$)

∎

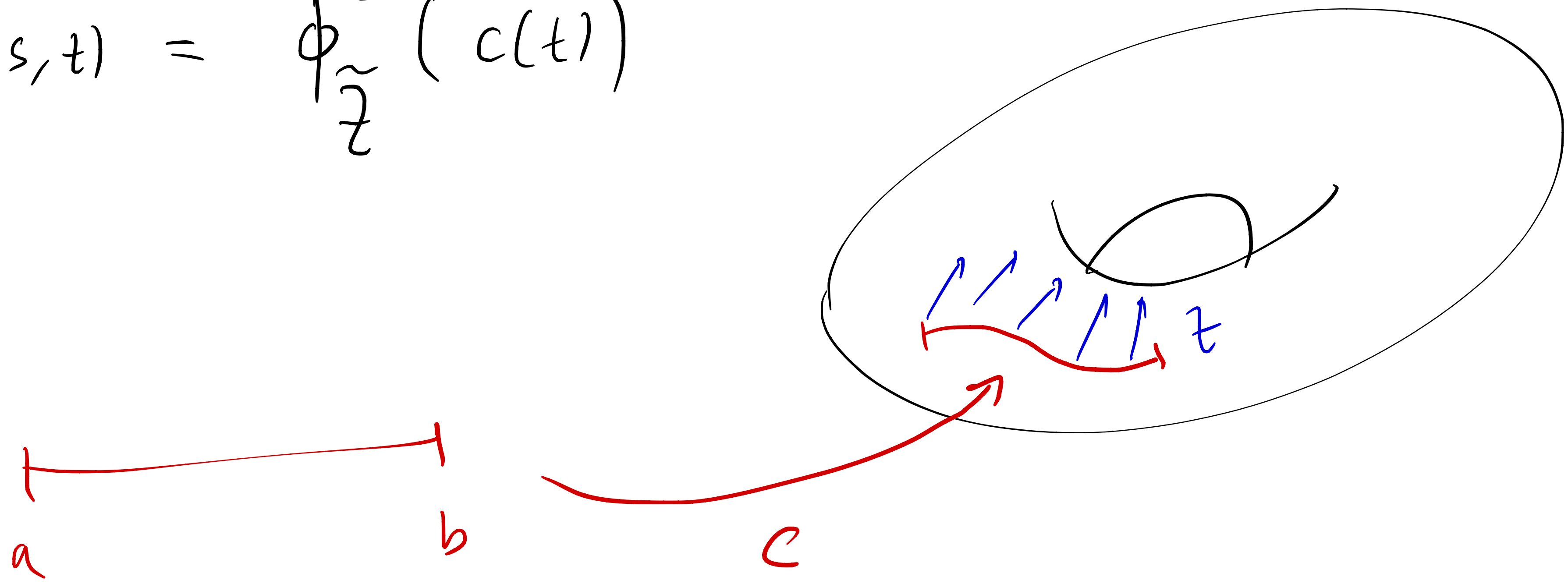
Two "practical" remarks $\qquad \underbrace{|c'| \neq 0}$

1st) Given $c:[a,b] \longrightarrow M$ regular smooth curve, injective

define $F: (-\varepsilon,\varepsilon)\times[a,b] \longrightarrow M$. Given some $Z \in \Gamma(c^*TM)$

extend $Z\circ c^{-1}$ to $\tilde{Z} : \Gamma(TM)$ s.t. $Z(t)$ l.i. $c'(t)$

$$F(s,t) = \phi^s_{\tilde{Z}}(c(t))$$



2nd) Recall $D_{A_i} A_j =: \sum_k \Gamma^k_{ij} A_k$, $\boxed{A_i = \frac{\partial}{\partial \varphi^i}}$

Thm 1.9 $\Longrightarrow$ $\Gamma^k_{ij} g_{k\ell} = \langle D_{A_i} A_j , A_k \rangle =$

$$= \frac{1}{2}\left(A_i g_{j\ell} + A_j g_{i\ell} - A_\ell g_{ij} \right)$$

$$\Rightarrow \quad \Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left(\frac{\partial}{\partial \varphi^i} g_{j\ell} + \frac{\partial}{\partial \varphi^j} g_{i\ell} - \frac{\partial}{\partial \varphi^\ell} g_{ij} \right)$$

(sum over ℓ)

<u>Prop. 1.16</u> (1) $\forall v \in TM$ $\exists!$ geodesic

$c_v : \underbrace{(\alpha_v, \omega_v)}_{\text{maximal interval of defin}} \longrightarrow M$ with $\dot{c}_v(0) = v$

(2) The set $W := \{ (v,t) : v \in TM,\ t \in (\alpha_v, \omega_v) \}$ is open subset of $TM \times \mathbb{R}$, and the map

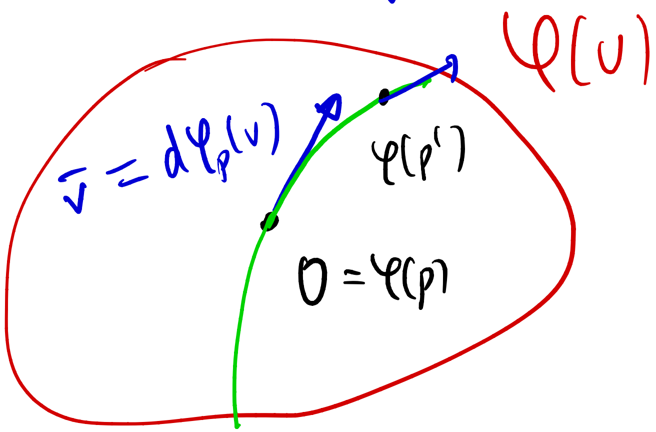
$(v,t) \longmapsto c_v(t)$ is C^∞

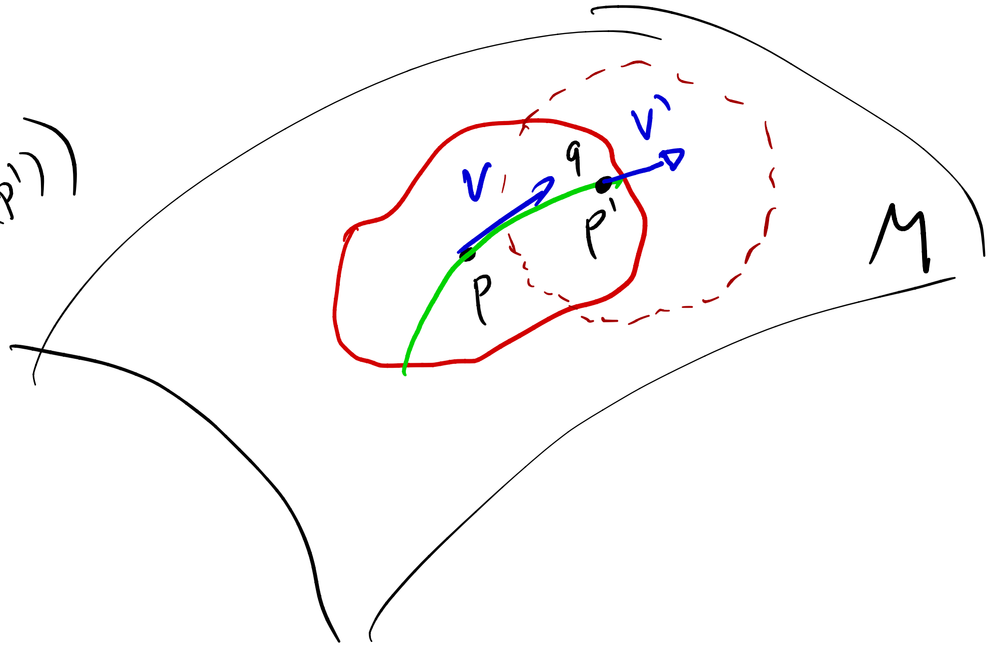
<u>proof</u> We want to reduce it to results of ODE theory in $\mathbb{R}^n$

→ The only difficulty is that $c_v((\alpha_v, w_v)) \not\supset U$ domain of a chart φ

$\bar{v}' = d\varphi_{p'}(v') = \dot{x}(t')$

$(x(t') = \varphi(p'))$





solve $\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i \dot{x}^j = 0$

$x(0) = 0, \quad \dot{x}(0) = v$

▨

Def'n the map $\tilde{W} \to M$, $\tilde{W} = \{ v \in TM : (v, 1) \in W \}$

length I am travelling along the geodesic

$$v \longmapsto c_v(1) = c_{\frac{v}{|v|}}(|v|)$$

unit vector at TM_p for some p

is called exponential map (exp)

Also the map $W \cap TM_p \to M$ is called exponential map at p and denoted $\exp_p$

$M = SO(n) \subset \mathbb{R}^{n \times n}$ Riem. mafld and group

$p = Id$ $A \in TM_p$ is the "Lie algebre" matrix B belong to $SO(n) \iff$

$\det(B) > 0$ and $B^T B = Id$

$A \in TM_p \iff A + A^T = 0$

$$B(t) = \exp(tA) = I + tA + \frac{t^2}{2} A^2 + \frac{t^3}{3!} A^3 + \dots$$

is the <u>geodesic</u> in $SO(n)$ with $B(0) = Id$
$B'(0) = A$

<u>Remark</u> $\underset{\Psi}{T(TM_p)_0} \simeq TM_p$

$$d(\exp_p)_0(w) = \frac{d}{ds}\Big|_{s=0} \overbrace{\exp_p(0+sw)}^{c_w(s)} = \dot{c}_w(0) = w$$

$$\Longrightarrow \quad d(\exp_p)_0 = \mathrm{id}$$

("inverse fcn thm" Chp 8 of DGI)

In particular 0 is regular pt. of $\exp_p$ $\Rightarrow$ $\exists V_p \subset \Omega_p$

where $\Omega_p := \{(v,1) \in W, \text{ st } v \in TM_p\}$ open nbhd of 0

such that $\exp_p\big|_{V_p} : V_p \longrightarrow \underbrace{\exp_p(V_p)}_{\overset{!!}{U_p}} \subset M$

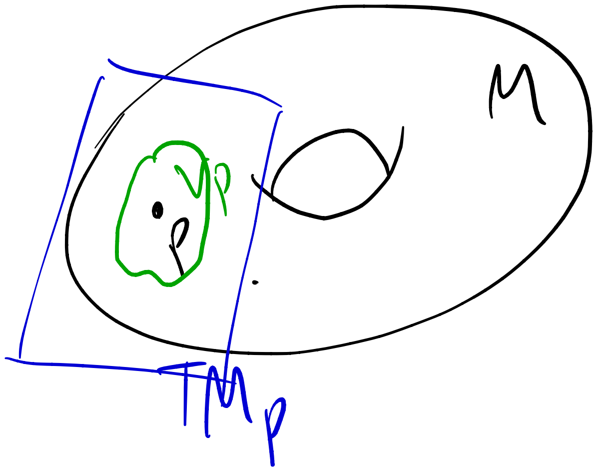
is a <u>diffeomorphism</u>.

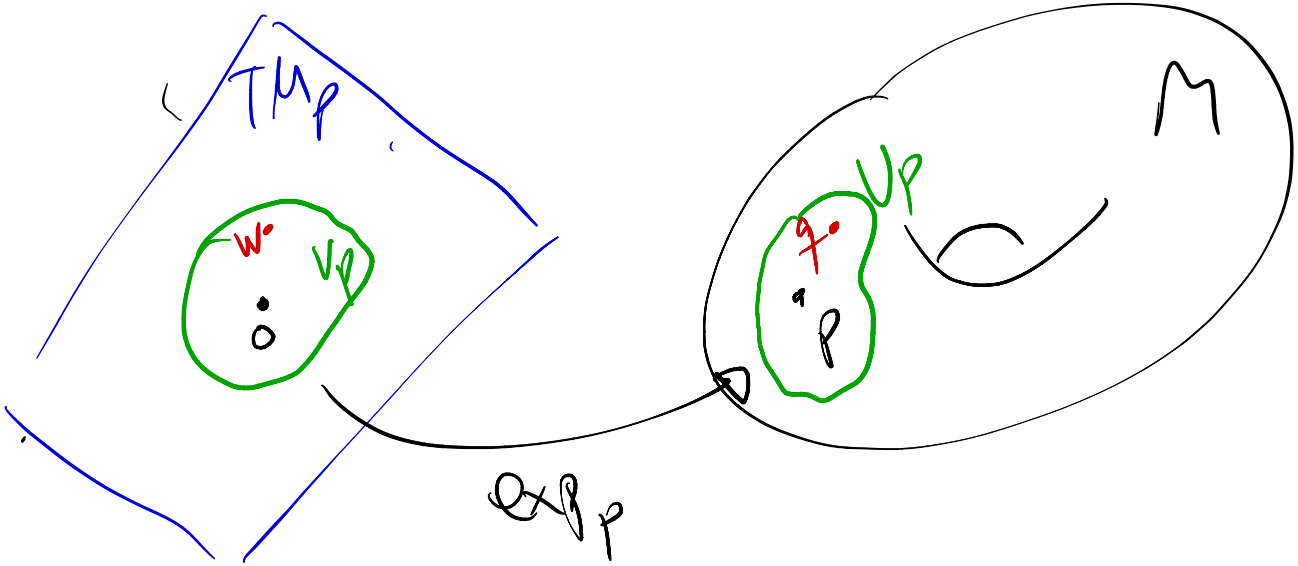
Therefore, for any "fixed" isometry

$$H : (TM_p, g_p) \longrightarrow (\mathbb{R}^m, \langle \cdot, \cdot \rangle)$$

we can define <u>normal coordinates</u> (at p)

$$\phi = H \circ \left(\exp_p\big|_{V_p}\right)^{-1} : \underset{\underset{M}{\cup}}{U_p} \longrightarrow H(V_p) \subset \mathbb{R}^m$$





__Observation__ $\quad H \longleftrightarrow$ choice of ONB of (TM_p, g_p)

$$\overline{e}_i = H^{-1}(e_i) \qquad e_i = (0, \ldots \overset{i}{1}, 0) \in \mathbb{R}^n$$

__Lemma 1.19__ In normal coordinates ϕ around p

$$\underbrace{g_{ij}(p) = \delta_{ij}}_{(1)} \qquad \underbrace{\frac{\partial g_{ij}}{\partial \phi^k}(p) = 0}_{(3)} \quad \text{and} \quad \underbrace{\Gamma^k_{ij}(p) = 0}_{(2)}$$

$$\left.\frac{\partial}{\partial \phi^i}\right|_p = d(\phi^{-1})_0(e_i) = H^{-1}(e_i) \quad \Leftarrow \quad ☺$$

$$g_{ij}(p) = \left\langle \left.\frac{\partial}{\partial \phi^i}\right|_p, \left.\frac{\partial}{\partial \phi^j}\right|_p \right\rangle_{g_p} = \left\langle H^{-1}(e_i), H^{-1}(e_j) \right\rangle_{g_p} =$$

$$\overset{H \text{ isom}}{=} \langle e_i, e_j \rangle_{\mathbb{R}^m} = \delta_{ij} \quad \Rightarrow \quad (1)$$

By def'n of normal coord. $t \mapsto c_{\bar v}(t)$ is mapped to $x(t) = \phi \circ c_{\bar v}(t) = \bar v t \qquad v = H(\bar v)$

$\Rightarrow \quad x(t) = v\,t$ is a geodesic for all $v \in \mathbb{R}^m$

$\Leftrightarrow$ (geodesic eq'n) $\ddot x^k + \Gamma^k_{ij} \dot x^i \dot x^j = 0 \quad (\forall k = 1, \dots, m)$

$\Leftrightarrow \quad \Gamma^k_{ij} v^i v^j = 0 \quad \Rightarrow \quad \Gamma^k_{ij} = 0$ at p.

(I have used that D is torsion free $\qquad \Rightarrow (2)$

$\Rightarrow \quad \Gamma^k_{ij} = \Gamma^k_{ji}$

To verify (3) use

D compatible $\Rightarrow$ $$\boxed{\frac{\partial g_{ij}}{\partial \phi^k} = \Gamma^{\ell}_{ik}\, g_{\ell j} + g_{i\ell}\, \Gamma^{\ell}_{kj}}$$

(exercise)

Prop 1.20 (Gauss' Lemma)

Given $v \in TM_p$ $\quad T_v := d(\exp_p)_v : \underset{\substack{\cong \\ T(TM_p)_v}}{TM_p} \longrightarrow TM_q$

$(q = \exp_p(v))$

$$\boxed{\langle T_v(v), T_v(w) \rangle_q = \langle v, w \rangle_p}$$

Consider $t \in [0,1]$

$$\gamma_s(t) = \exp_p\big(t\,\underbrace{(v+sw)}\big) = \text{geodesic with speed } |v+sw|$$

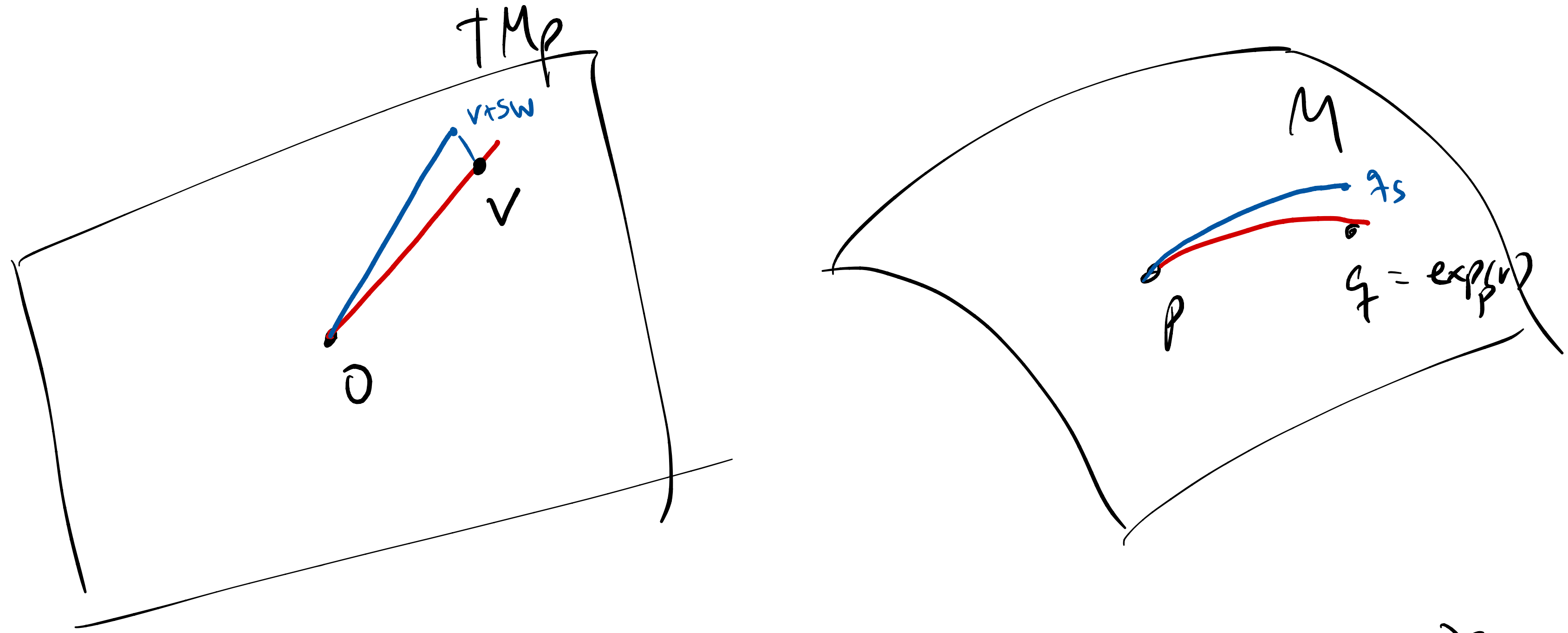
$$\gamma : [0,\underline{\underline{1}}] \longrightarrow M$$

[1st variation of length (Thm 1.15) + $\exp(tv)$ is geodesic]

$$\frac{d}{ds}\Big|_{s=0} L(\gamma_s) = \frac{1}{|v|}\Big\langle \underbrace{\frac{d}{ds}\Big|_{s=0}\gamma_s(\underline{\underline{1}})}_{V_0(1)}, \ \dot{c}_v(\underline{\underline{1}}) \Big\rangle_{c_v(\underline{1})}$$

(Notice $V_0(0) = 0$)

$$= \frac{1}{|v|}\big\langle T_v(w), T_v(v)\big\rangle_q$$



By defn of $\exp_p$, since γ_s is a geodesic $\gamma_s(0) = p$:

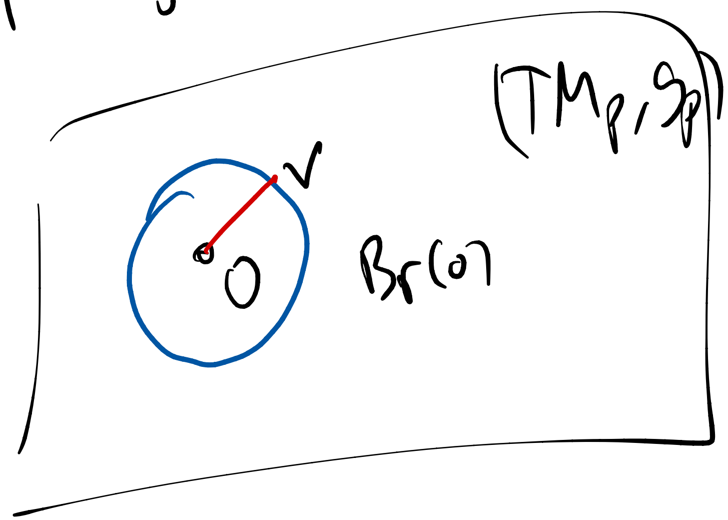
$$L(\gamma_s) = |v+sw|_{g_p} = \sqrt{\langle v+sw, v+sw\rangle_{g_p}}$$

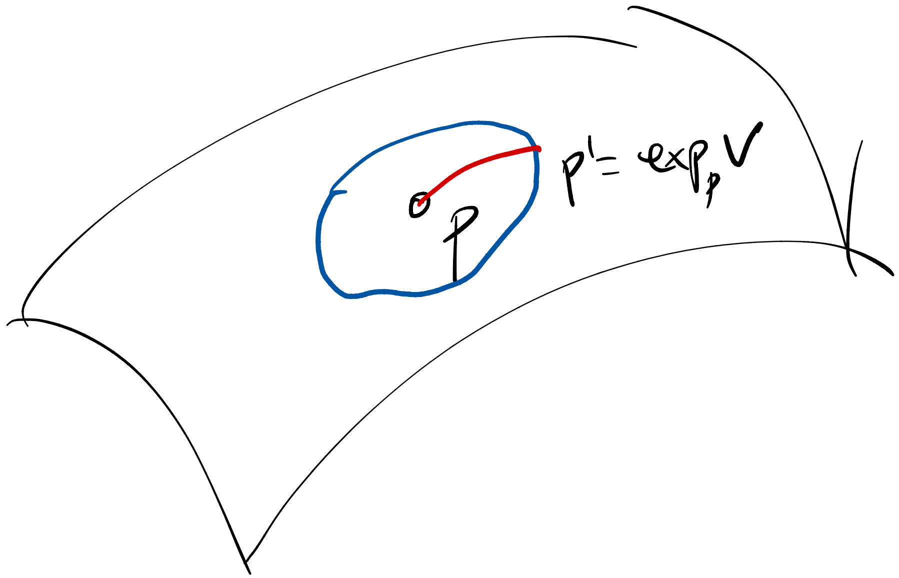
$$\frac{d}{ds}\Big|_{s=0} L(\gamma_s) = \frac{1}{\cancel{2}|v|}\cancel{2}\langle w, v\rangle_p$$

∎

Prop 1.21 $p \in M$, let $\tilde{r} > 0$ s.t. $\exp_p \big|_{B_{\tilde{r}}(0)}$ is diffeomorphism, and $r \in (0, \tilde{r})$

(1) every ray joining 0 and $\partial B_r(0)$ is mapped by $\exp_p$ onto a length-minimizing geodesic (of length r)





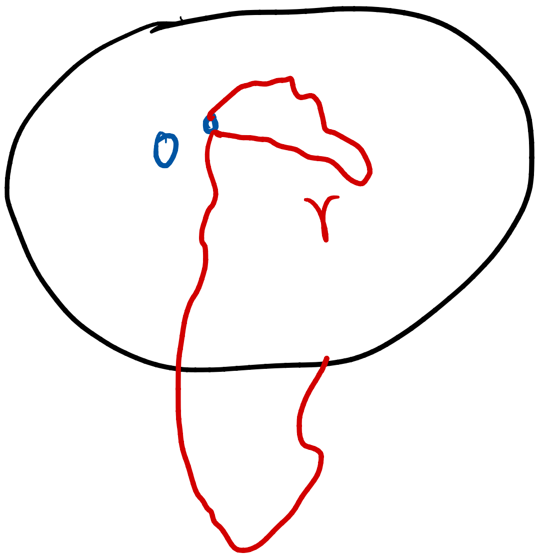
pf (1) we need to show that $\forall \gamma : [a,b] \longrightarrow M$

$\gamma(a) = p$ and $\gamma(b) \in \exp_p(\partial B_r)$

$$L(\gamma) \geq r$$

Assume w.l.o.g $\gamma\big|_{(a,b)}$ has image contained in $\exp_p(B_r) - \{p\}$

(Indeed:



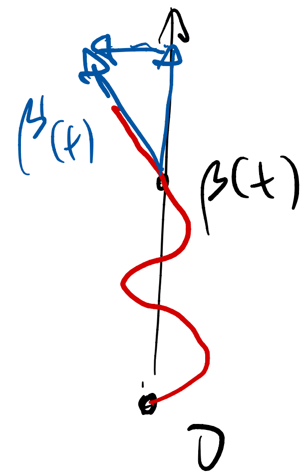
$\exp_p(\partial B_r)$)

$$\beta(t) := \exp_p^{-1}(\gamma(t)) \in TM_p \iff (\gamma = \exp_p \circ \beta)$$

$$\beta(a) = 0, \quad |\beta(b)| = r \quad (\iff \beta(b) \in \partial B_r)$$

$$\beta((a,b)) \subset B_r \setminus \{0\}$$

$\langle \; \rangle_p$



$$\beta'(t) = \lambda(t)\beta(t) + w(t)$$

$$\text{s.t. } \langle w, \beta \rangle \equiv 0$$

$$\langle \beta'(t), \beta(t) \rangle \overset{*}{=} \lambda(t)|\beta(t)|^2$$

$$\gamma'(t) = T_{\beta(t)}(\beta'(t))$$

where T as is Gauss' Lemma

$$|\gamma'(t)|^2_{\gamma(t)} \overset{\text{Gauss' Lemma}}{=} (\lambda(t))^2 |\beta(t)|^2 + \underbrace{|T_{\beta(t)}(w(t))|^2}_{\geq 0}$$

$$L(\gamma) = \int_a^b |\gamma'(t)|\, dt \geq \int_a^b |\lambda(t)\,\beta(t)|\, dt$$

$$\geq$$

Recall $\beta(b) \in \partial B_r$, $\beta(a) = 0$

$$r = |\beta(b)| = |\beta(b)| - |\beta(a)| = \int_a^b |\beta(t)|'\, dt$$

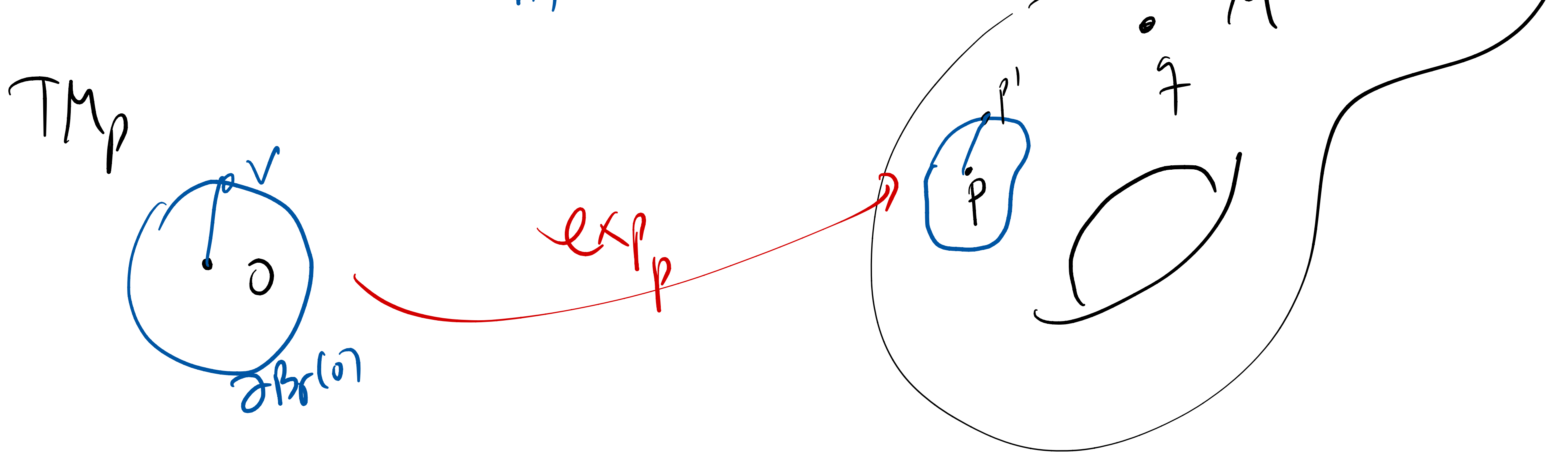
$$|\beta(t)|' = \frac{d}{dt}\sqrt{\langle \beta(t), \beta(t)\rangle} = \frac{1}{2|\beta(t)|}\, 2\langle \beta'(t), \beta(t)\rangle$$

$$\overset{*}{=} \lambda(t)|\beta(t)|$$

Prop 1.21 (2) $\forall q \in M \setminus \exp_p(\overline{B_r(0)}) \quad \exists v \in \partial B_r(0)$

triangle inequality

$$d(p,q) \overset{\leq}{=} \underset{\substack{\| (1) \\ d(p,p')}}{r} + d(\underbrace{\exp_p(v)}_{p'}, q)$$



I need to show $\exists p' \in \exp_p(\partial B_r(0))$ s.t. "reversed" triangle ineq. holds $\geq$

Choose γ almost achieving the $d(p,q)$

$\forall \varepsilon > 0 \ \exists \gamma : \quad \gamma(a) = p, \ \gamma(b) = q$

$$L(\gamma) \leq d(p,q) + \varepsilon$$

$$t_* = \inf \{ t \in (a,b) : \gamma(t) \notin \exp_p(B_r) \}$$

$$d(q,p) + \varepsilon \geq L(\gamma) = L(\gamma|_{[a,t_*]}) + L(\gamma|_{[t_*,b]})$$

$$\overset{(1)}{\geq} d(p, \exp_p(v)) + d(\exp_p(v), q)$$

$v = \exp_p^{-1}(\gamma(t_*))$ <u>because</u> $\gamma(t_*) \in \exp_p(\partial B_r)$

$p' = \exp_p(v)$ take $\varepsilon \downarrow 0$

Thm (-def'n) Hopf-Rinow (M, g) connected Riem. mfld

The following are equivalent:

(1) (M, d) is complete (as metric space, i.e. Cauchy seq. are convergent)

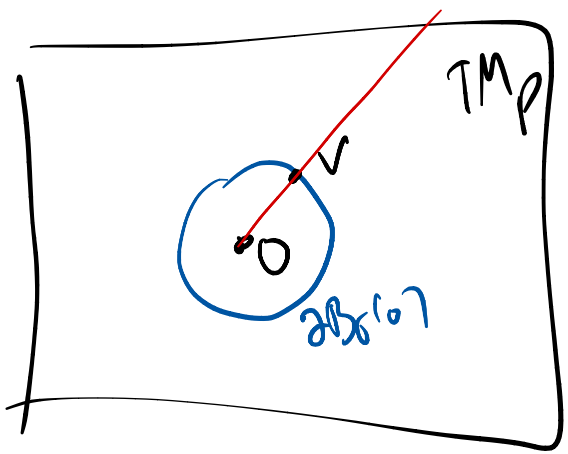
(2) (M, d) is <u>geodesically complete</u>: exp is defined on all of TM

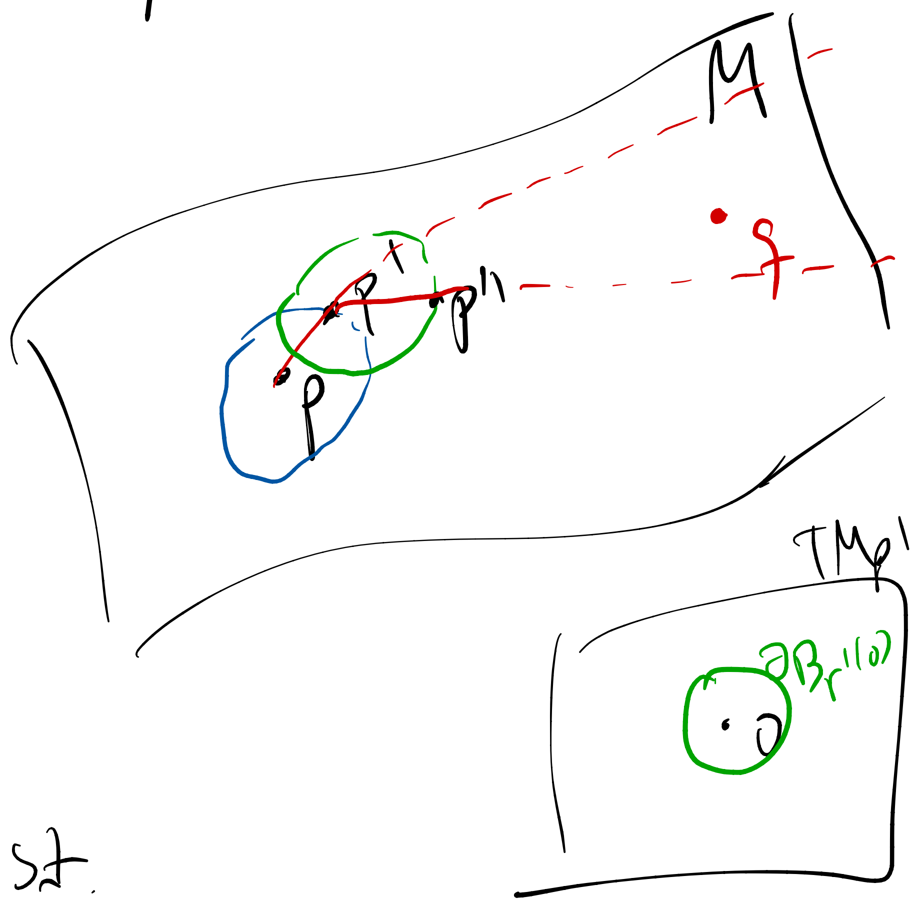
(3) $\exists\, p \in M$ s.t. $\exp_p$ is defined on all of TM_p

(4) Bdd + closed subsets of M are compact

When this happens we say (M,g) is <u>complete</u>

<u>proof</u> $(3) \Longrightarrow$ ✪ $\forall q \in M\ \exists$ geodesic of length $d(p,q)$ joining p and q





By prop 2.21 (b) $\exists\ p' = \exp_p(\overset{\partial B_\delta(0)}{\overset{\cup}{v}})$ s.t.

$$d(p,q) = d(p,p') + d(p',q)$$

Let $c_v(t) = \exp_p(vt)$ geodesic ray emanating from p

$$d(p, c_v(t)) + d(c_v(t), q) = d(p,q) \qquad ☺$$

☺ is satisfied for $t \leq r$

So, let $t_* := \sup \{ t \in (0, d(p,q)] \text{ s.t. ☺ holds} \}$

(I want to show $t_* = d(p,q)$)

Suppose by cont, $t_* < d(p,q)$, let $p_* := c_v(t_*)$

$\exists\, r_* > 0$ st $\exp_{p_*}\big|_{B_{r_*}(0)}$ is diffeo

Hence by prop 121 (2) $\exists\ p_*' \in \exp_{p_*}(\partial B_{r_*}(0))$

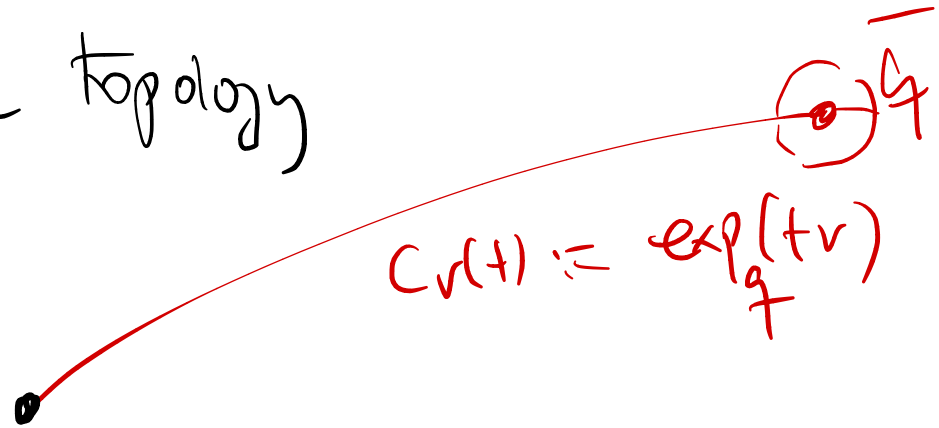
$$d(p_*, q) = d(p_*, p_*') + d(p_*', q)$$

$\Rightarrow$ ☺ holds for $t > t_*$ sufficiently close to t_* !!

★ $\Rightarrow$ (4)   Indeed, $M \subset \bigcup_{r>0} \exp_p(\overline{B_r})$

(4) $\Rightarrow$ (1)   Basic topology

(1) $\Rightarrow$ (2)



Fix $q \in M$, $v \in TM_q$, $|v| = 1$

Say it is defined only for $t \in [0, t_*)$ $\quad t_* < +\infty$

Whenever $t_k \uparrow t_*$

$$d(c_v(t_k), c_v(t_m)) \leq |t_k - t_m|$$

$\Rightarrow$ $\quad q_k := c_v(t_k)$ is a Cauchy sequence

$$q_k \to \bar{q}$$

But then $\exp_{\bar{q}}$ is defined in nbhd of 0

so $c_v(t)$ can be continued past $t = t_*$

(2) $\Rightarrow$ (3) Obvious (implicitly uses $M \neq \emptyset$) $\blacksquare$

Riemannian curvature

Riem. mfld (M,g) with Levi-Civita connection D

$$R : \Gamma(TM)^3 \longrightarrow \Gamma(TM)$$

$$R(X,Y)W := D_X D_Y W - D_Y D_X W - D_{[X,Y]} W$$

" $R(X,Y) = [D_X D_Y] - D_{[X,Y]}$ "

is Riem. curvature tensor

$R(X,Y)$ (for X, Y fixed) $\mathbb{R}$-linear map

from $\Gamma(TM) \longrightarrow \Gamma(TM)$

Observe $R(X,Y) \equiv -R(Y,X) \Leftrightarrow (\forall W,\ R(X,Y)W = -R(Y,X)W)$

<u>R is tensor</u>

$$R(fX,Y)W = D_{fX}D_Y W - D_Y D_{fX} W - D_{\underbrace{[fX,Y]}_{f[X,Y]-Y(f)X}} W$$

$$= f D_X D_Y W - D_Y (f D_X W)$$

$$- D_{f[X,Y]-Y(f)X} W$$

$$= f D_X D_Y W - f D_Y D_X W - Y(f) D_X W$$

$$- D_{f[X,Y]} W + D_{Y(f)X} W$$

$$= f R(X,Y) W$$

$\Rightarrow$ R is tensorial w.r.t X (and also w.r.t Y by antisymmetry)

<u>Exercise</u> $R(X,Y)(fW) = f R(X,Y) W$

How is the Riemann tensor written in local coordinates?

Let (ϕ, U) be a chart $\qquad A_i = \dfrac{\partial}{\partial \phi^i}$

$$D_{A_k} D_{A_l} A_j = D_{A_k}\left(\Gamma^s_{lj} A_s\right)$$

(Einstein's summation conv)

$$= A_k(\Gamma^i_{lj}) A_i + \Gamma^s_{lj} \Gamma^i_{ks} A_i$$

Define R^i_{jkl} by $R(A_k, A_l) A_j =: R^i_{jkl} A_i$

$[A_i, A_j] = 0 \qquad A_k = \frac{\partial}{\partial \phi^k}$

Therefore $R^i_{jkl} = \frac{\partial}{\partial \phi^k}\left(\Gamma^i_{lj}\right) - \frac{\partial}{\partial \phi^l}\left(\Gamma^i_{kj}\right)$

$$+ \left(\Gamma^s_{lj} \Gamma^i_{ks} - \Gamma^s_{kj} \Gamma^i_{ls}\right)$$

We can also define R as a $(0,4)$ tensor field

$$R(V,W,X,Y) = \langle V, R(X,Y)W \rangle$$

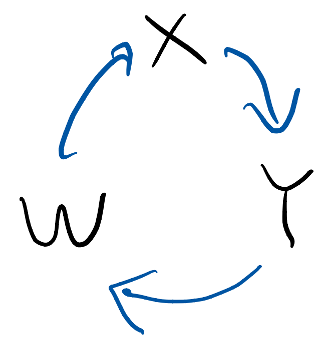
$$\begin{aligned} R_{ijkl} &:= R(A_i, A_j, A_k, A_l) \\ &= \langle A_i, R(A_k, A_l) A_j \rangle = \langle A_i, R^r_{jkl} A_r \rangle \\ &= R^r_{jkl} \langle A_i, A_r \rangle \\ &= g_{ir} R^r_{jkl} \end{aligned}$$

__Prop 2.2__ (symmetries of R)

(1) $$R(Y,X)W = -R(X,Y)W \iff R^i_{jkl} = -R^i_{jlk}$$

$$\iff R(V,W,Y,X) = -R(V,W,X,Y)$$

(2) $$\sum_{\substack{(X,Y,W)\\ \text{cyclic}}} R(X,Y)W = 0$$



__Rem__ It is enough to prove the identities

$$X, Y, W \in \{A_i\} \quad A_i = \frac{\partial}{\partial \phi^i} :$$

So, we can assume w.l.o.g Lie Brackets $= 0$

$$R(X,Y)W + R(Y,W)X + R(W,X)Y =$$

$$= D_X D_Y W - D_Y D_X W + D_Y D_W X - D_W D_Y X + D_W D_X Y - D_X D_W Y$$

D is torsion free!

$$= 0$$

(3) $R(W,V,X,Y) = -R(V,W,X,Y)$

Notice: for fixed X, Y

$B(V,W) := R(V,W,X,Y)$ is a bilinear form

Any bilinear form B satisfies the

$$(PI) \quad 2\big(B(V,W) + B(W,V)\big) = B(W+V, W+V) - B(W-V, W-V)$$

Therefore, $(3) \iff R(V,V,X,Y) \equiv 0$

$$\iff \langle V, R(X,Y)V \rangle \equiv 0 \qquad \text{(exercise)}$$

To show this use D compatible

$$XY\langle V,V\rangle = X\big(2\langle D_Y V, V\rangle\big)$$
$$= 2\langle D_X D_Y V, V\rangle + 2\langle D_Y V, D_X V\rangle$$

(4) $\quad R(X, Y, V, W) = R(V, W, X, Y)$

Follows from (1)-(2)-(3) with "trick" from notes

SUMMARY OF SYMMETRIES OF R $\quad$ (0,4) form

$$R(V, W, X, Y)$$

Observation

1. $B^V(W,Y) = R(V,W,V,Y)$ symmetric bilinear form

2. the values of $R(V,W,V,Y)$ for all $V,W,Y \in \Gamma(TM)$ completely determine $R(V,W,X,Y)$ (exercise)

3. $B^V(W,Y)$ is completely determined by $B^V(W,W)$

R is determined by $\underbrace{R(V,W,V,W)}$

$$R(sv+tw, rw, sv+tw, rw) =$$

$$= r^2 s^2 R(v,w,v,w) + \cancel{t R(w,w,v,w)}$$

$$+ \cancel{t R(v,w,w,w)} + \cancel{t^2 R(w,w,w,w)}$$

$v, w \in TM_p$

$$\Longrightarrow \quad R(v,w,v,w) = \lambda\left(|v|^2|w|^2 - \langle v, w\rangle^2\right)$$

λ only depends on the 2-plane $P \subset TM_p$ generated by v, w

<u>Def'n</u> Sectional curvature for any 2-plane $P \subset TM_p$

$$\sec(P) := \frac{R(v,w,v,w)}{|v|^2|w|^2 - \langle v,w\rangle^2} \quad (= \lambda)$$

for some v, w spanning P

Rem. We showed that sec $\xrightarrow{\text{determines}}$ R

Def'n 2.6 $\exists \kappa \in \mathbb{R}$ $\quad sec(P) \equiv \kappa$ $\quad \forall p \in M$, $\forall P$ 2-plane $\subset T M_p$

space of constant sec. curv.

$$\underset{\text{exercise}}{\Longrightarrow} \quad R(X,Y)W = \kappa\left(\langle Y,W\rangle X - \langle X,W\rangle Y\right)$$

A goal of the course

M complete, simply connected with cst. sec. curv $\equiv \kappa$

$$M = \begin{cases} \text{sphere} & \kappa > 0 \\ \text{Euclidean space} & \kappa = 0 \\ \text{Hyperbolic space} & \kappa < 0 \end{cases}$$

On tensor fields ...

In local coordinates (U,ϕ), a (r,s)-tensor field T has a local representation

$$T|_U = T^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_r} \frac{\partial}{\partial \phi^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial \phi^{i_r}} \otimes d\phi^{j_1} \otimes \cdots \otimes d\phi^{j_s}$$

Different coordinat $\tilde{\phi} : U \longrightarrow \mathbb{R}^m$

$$\left.\begin{aligned} \frac{\partial}{\partial \phi^{i_r'}} &= \frac{\partial \tilde{\phi}^{\alpha}}{\partial \phi^{i_r'}} \frac{\partial}{\partial \tilde{\phi}^{\alpha}} \\ d\phi^{i_s'} &= \frac{\partial \phi^{i_s'}}{\partial \tilde{\phi}^{\beta}} d\phi^{\beta} \end{aligned}\right\} \quad \text{"chain rule"}$$

$$R|_U = R^i_{jkl} \frac{\partial}{\partial \phi^i} \otimes d\phi^j \otimes d\phi^k \otimes d\phi^l \qquad (*)$$

How R^i_{jkl} transform when choosing new coord $\tilde{\phi}$?

$$\frac{\partial}{\partial \phi^i} = \frac{\partial \tilde{\phi}^\alpha}{\partial \phi^i} \frac{\partial}{\partial \tilde{\phi}^\alpha} \qquad d\phi^j = \frac{\partial \phi^j}{\partial \tilde{\phi}^\beta} d\tilde{\phi}^\beta$$

resp. k, l

resp γ, δ

$$R|_U = \underbrace{\tilde{R}^\alpha_{\beta\gamma\delta}}_{?} \frac{\partial}{\partial \tilde{\phi}^\alpha} \otimes d\tilde{\phi}^\beta \otimes d\tilde{\phi}^\gamma \otimes d\tilde{\phi}^\delta$$

$$R|_U = R^i_{jkl} \left(\frac{\partial \tilde\phi^\alpha}{\partial \phi^i} \frac{\partial}{\partial \tilde\phi^\alpha} \right) \otimes \left(\frac{\partial \phi^j}{\partial \tilde\phi^\beta} d\tilde\phi^\beta \right) \otimes \left(\frac{\partial \phi^k}{\partial \tilde\phi^\gamma} d\tilde\phi^\gamma \right) \otimes \left(\frac{\partial \phi^l}{\partial \tilde\phi^\delta} d\tilde\phi^\delta \right)$$

$$= \underbrace{\left(R^i_{jkl} \frac{\partial \tilde\phi^\alpha}{\partial \phi^i} \frac{\partial \phi^j}{\partial \tilde\phi^\beta} \frac{\partial \phi^k}{\partial \tilde\phi^\gamma} \frac{\partial \phi^l}{\partial \tilde\phi^\delta} \right)}_{\parallel \atop \tilde R^\alpha_{\beta\gamma\delta}} \frac{\partial}{\partial \tilde\phi^\alpha} \otimes d\tilde\phi^\beta \otimes d\tilde\phi^\gamma \otimes d\tilde\phi^\delta$$

Contractions (trace) T is $(1,1)$ tensor field in M^m

$$(\mathrm{trace}(T))(p) = \sum_{j=1}^m T_p(\varepsilon^j, e_j)$$

e^j is a basis of TM_p and ε^j is the associated dual basis

This gives us a smooth function

Similarly for $(4,3)$ tensor field T we can do contraction

$$C^3_2 \, T = \sum_{j=1}^{m} T(\cdot, \cdot, \varepsilon^j, \cdot, \cdot, e_j, \cdot)$$

$$T(\omega_1, \omega_2, \underbrace{\omega_3}_{3}, \omega_4, X_1, \underbrace{X_2}_{2}, X_3)$$

In coordinates (U, ϕ) a $(1,1)$ tensor field is

$$T|_U = T^i_j \frac{\partial}{\partial \phi^i} \otimes d\phi^j$$

$$\mathrm{trace}(T)|_v = T^i_{\ i}$$

Also in a Rien. mfld (M,g) we can consider

<u>metric contractions</u>

<u>Example</u> $(0,4)$ tensor field $\rightsquigarrow$ "see it" as $(1,3)$ tensor field (using g)

and then contract the superindex with one of the subindices

$$T(X_1, X_2, X_3, X_4) = \langle \tilde{T}(X_1, X_2, X_3), X_4 \rangle$$

$$C_{14} T|_p = \varepsilon^j\left(\tilde{T}(e_j|_p, X_2|_p, X_3|_p)\right)$$

Equivalently, choose ONB E_i of (TM_p, g_p)

$$C_{14} T|_p = \sum_{i=1}^{m} T(\underbrace{E_i}_{1}, \cdot, \cdot, \underbrace{E_i}_{4})$$

In coordinates:

$$T_{\underbrace{i}_{1} jk \underbrace{l}_{4}} \, d\phi^i \otimes d\phi^j \otimes d\phi^k \otimes d\phi^l$$

$$C_{14} T = g^{il} T_{ijkl} \, d\phi^j \otimes d\phi^k$$

[side comment / example]

x, y coordinates of plane

v, w new coordinates

$v = \sin(x + y^2)$

$w = y$

$dv = \cos(x+y^2)(dx + 2y\,dy)$

$dw = dy$

$$dv \wedge dw = \cos(x+y^2)\, dx \wedge dy$$

Ricci tensor is the metric contraction of R

$$\mathrm{Ric}(v,w)|_p = \sum_{i=1}^{m} R(e_i, v, e_i, w)|_p$$

$\parallel$ symmetries of R

$\mathrm{Ric}(w,v)$

where $e_i|_p$ is ONB of (TM_p, g_p).

In coordinates,

$$\mathrm{Ric}|_U = \underbrace{R_{jl}}\, d\phi^j \otimes d\phi^l$$

$$g^{ik} R_{ijkl}$$

Scalar curvature

e_j ONB of (TM_p, g_p)

$$\mathrm{Scal}(p) := \sum_{j=1}^{m} \mathrm{Ric}(e_j, e_j)\Big|_p$$

So, $$\mathrm{scal}|_U = g^{ik} R_{ik} = g^{ik} g^{jl} R_{ijkl}$$

Exercise If (M^m, g) has constant sec. curv $\equiv K$

$$\mathrm{Ric} \equiv (m-1) K g \qquad \mathrm{scal} \equiv m(m-1) K$$

Grad, div, and Laplace $\quad (M, g)$ Riem. mfld

$f \in C^\infty(M)$

$\operatorname{grad} f \in \Gamma(TM)$ s.t. $\forall X \in \Gamma(TM)$

$$\langle \operatorname{grad} f, X \rangle = df(X) = Xf$$

$$\begin{aligned}
\operatorname{Hess} f(X, Y) &= \langle D_X \operatorname{grad} f, Y \rangle \\
&= X\langle \operatorname{grad} f, Y \rangle - \langle \operatorname{grad} f, D_X Y \rangle \\
&= X(df(Y)) - df(D_X Y)
\end{aligned}$$

<u>Exercise</u> Hess f is symetric

<u>Covariant different.</u> of (i.e. Levy-Civita connection acting on) tensor fields

We know: $X, Y \in \Gamma(TM)$, $f \in C^\infty(M)$

$D_X Y$ ✓

$D_X f := Xf$ ✓

What happens with a 1-form $\omega \in \Gamma(TM^*)$

I wish it was true:

$$D_X(\omega(Y)) = X(\omega(Y))$$
$$= (D_X\omega)(Y) + \omega(D_X Y)$$

So, let me take as def'n of $D_X\omega$

$$(D_X\omega)(Y) := X(\omega(Y)) - \omega(D_X Y)$$

More in general: T is (r,s)-tensor field

$$(\omega_1, \dots, \omega_r, X_1, \dots, X_s) \longmapsto \text{some } C^\infty(M) \text{ fcn}$$

(multilinear, $C^\infty(M)$ homog.)

I wish: $\forall Y \in \Gamma(TM)$

$$Y\big(T(\omega_1, \dots, \omega_r, X_1, \dots, X_s)\big) =:$$

$$(D_Y T)(\omega_1, \dots, \omega_r, X_1, \dots, X_s) + T(D_Y \omega_1, \omega_2, \dots, X_s)$$

$$\vdots$$

$$+ T(\omega_1, \dots, D_Y \omega_r, X_1, \dots, X_s)$$

$$+ T(\omega_1, \dots, \omega_s, D_Y X_1, \dots, X_s)$$

$$\vdots$$

$$+ T(\omega_1, \dots, \omega_s, X_1, \dots, D_Y X_s)$$

__Remark.__ Notice $(D_Y T)$ is $C^\infty(M)$-homog. w.r. to all of its variables, hence a tensor

$Y \in \Gamma(TM)$

$\operatorname{div} Y$ is the contraction of the $(1,1)$ tensor $D_\bullet Y$

Exercise: write div in coordinates

$f \in C^\infty(M)$,

Laplace-Beltrami operator $\Delta f = \operatorname{div}(\operatorname{grad} f)$

Remarks 1. $D_Z(C^i_j T) = C^i_j(D_Z T)$ see concrete example below

2. T is $(1,s)$ tensor $\longleftrightarrow$ map "eating" s vector fields and "spitting" 1 vector field

Check consistency of the following def'n of $D_Z T$ with previous one:

$$
\begin{aligned}
(D_Z T)(X_1, \dots X_s) = \; & D_Z(T(X_1, \dots, X_s)) \\
& - T(D_Z X_1, \dots, X_s) \\
& \vdots \\
& - T(X_1, \dots, D_Z X_s)
\end{aligned}
$$

Next goal

<u>Prop 2.15</u> (M^m, g) Rien. mfld

$$d\, \mathrm{Scal} = 2\, \mathrm{div}(\mathrm{Ric})$$

$$\mathrm{div}(\mathrm{Ric}) = \text{metric contraction of } D_\bullet \mathrm{Ric}$$

$$= \sum_{i=1}^{m} (D_{E_i} \mathrm{Ric})(E_i, \cdot)$$

$(E_i \text{ ONB})$

To prove it I need Rem 1 above:

T is a $(0,2)$ tensor field

$$C(T)_p = \sum_{i=1}^{m} T(E_i, E_i) \qquad E_i|_p \text{ is ONB}$$

$C(T)$ belongs to $C^\infty(M)$

$$D_X(C(T)) = X(C(T)) \overset{?}{=} C(D_X T) \quad (\text{at } p)$$

$$= \sum_{i=1}^{m} \underbrace{(D_X T)}(E_i, E_i)$$

By choosing a curve

$c : (-\varepsilon, \varepsilon) \longrightarrow M$

$c(0) = p$, $\dot{c}(0) = X$

and take E_i any

extension of $E_i|p$ s.t.

$E_i \circ c$ is parallel (along c)

$$= \sum_{i=1}^{m} \Big(X(T(E_i, E_i)) - T(\cancel{D_X E_i}, E_i) - T(E_i, \cancel{D_X E_i}) \Big)$$

$$= X\left(\sum_{i=1}^{m} T(E_i, E_i)\right)$$

Another needed ingredient is 2^{nd} Bianchi id

Lemma 2.13

$$\sum_{\substack{\text{cyclic} \\ (X,Y,Z)}} (D_Z R)(X,Y) = 0$$

(i.e $(D_Z R)(X,Y)W + (D_X R)(Y,Z)W + (D_Y R)(Z,X)W = 0$

for all $X, Y, Z, W \in \Gamma(TM)$)

proof

$$D_Z(R(X,Y)W) = (D_Z R)(X,Y)W$$

$$+ R(D_Z X, Y)W + R(X, D_Z Y)W + R(X,Y)D_Z W$$

it is enosgh $X, Y, Z, W \in \left\{ \frac{\partial}{\partial \phi^i} \right\}_{i=1,\dots,m}$

$$(D_Z R)(X,Y)W = \underbrace{\overbrace{D_Z [D_X, D_Y] W}^{D_Z D_X D_Y W - D_Z D_Y D_X W}}_{1}$$

$$- \underbrace{R(D_Z X, Y)W}_{2} + \underbrace{R(\underbrace{D_Z Y}_{D_Y Z}, X)W}_{2'} \overbrace{- \underbrace{[D_X, D_Y] D_Z W}_{-(D_X D_Y D_Z W - D_Y D_X D_Z W)}}^{1'}$$

when I take a cyclic sum (X, Y, Z)

1 cancels 1'

2 cancles 2'

$\blacksquare$

proof of Prop 7.15

$$R(V,W,X,Y) = -R(W,V,X,Y)$$

Start with 2nd Bianchi id:

$$\langle V, D_Z R(X,Y)W\rangle + \langle V, D_X R(Y,Z)W\rangle + \langle V, D_Y R(Z,X)W\rangle = 0$$

$$\langle V, D_Z R(X,Y)W\rangle = \langle W, D_X R(Y,Z)V\rangle + \langle W, D_Y R(Z,X)V\rangle$$

$$V, W, X, Y \Big|_p \xleftarrow{\text{replace}} e_i \; e_j \; e_i \; e_j \qquad \underline{\text{ONB}}$$

$$\underbrace{\sum_{i,j=1}^{m} \langle e_i, (D_Z R)(e_i, e_j) e_j\rangle}_{\text{!! LHS}} = 2 \sum_{i,j=1}^{m} \langle e_i, (D_{e_j} R)(e_i, Z) e_j\rangle$$

$$= 2 \sum_{i,j=1}^{m} D_{e_j}\big(\langle e_i, R(e_i, Z) e_j\rangle\big)$$

$$LHS = D_z\left(\sum_{i,j=1}^{m} \langle e_i, R(e_i,e_j)e_j\rangle\right) \quad = 2\sum_{j=1}^{m} D_{e_j} Ric(z, e_j)$$

$$= D_z(scal) \quad = 2 \operatorname{div} Ric(z)$$

$$= d\,scal(z)$$

$\blacksquare$

Thm 2.16 (Schur 1886) (M,g) connected Riem. mfld of dim $m \geq 3$, then:

(1) $Ric = f\, g$ for some $f \in C^\infty(M)$

$\Longrightarrow f \equiv ct$

(2) If $\forall p \in M$ $sec(P)$ is the same for all planes $P \subset T M_p$

then sec is ctt on M

e_i ONB

<u>proof</u> (1) $\mathrm{Ric} = fg \Rightarrow \mathrm{scal} = \sum_{i=1}^{m} f\, g(e_i, e_i)$

$= mf$

using Prop 2.15

$$m\, df = d\,\mathrm{scal} = 2\,\mathrm{div}(\mathrm{Ric})$$

$$= 2\,\mathrm{div}(fg) = 2\,df$$

(Exercise check $\mathrm{div}(fg) = df \quad \forall f \in C^\infty(M)$)

$\Rightarrow \quad df \equiv 0 \quad \Rightarrow \quad f \equiv \mathrm{ctt}$

(2) $\sec(P) = K_p \quad \forall P \subset T M_p$ 2-plane

$\Longrightarrow \quad Ric_p = (m-1) K_p g$ apply (1) ∎

exercise

$$Ric(V,V) = \sum_{i=1}^{m} R(e_i, V, e_i, V)$$

e_i ONB st $e_1 = \frac{V}{|V|}$

$$= \Big(R(e_1, e_1, e_1, e_1) + \sum_{i=2}^{m} \underbrace{R(e_i, e_1, e_i, e_1)}_{K_p} \Big) |V|^2$$

$$= (m-1) K_p |V|^2$$

exercice $(D_Z g) \equiv 0 \quad \forall Z \in \Gamma(TM)$

Curvature of submanifolds $(M \subset \bar{M})$

- M m-dim submanifold of $\bar{M}$ $\bar{m}$ dim manifold
- $\bar{g}$ metric on $\bar{M}$

$\bar{g}$ induces a metric g on M $\qquad TM_p \subset T\bar{M}_p$

$$g(X,Y) := \bar{g}(\tilde{X},\tilde{Y}) \qquad \text{for all } \tilde{X},\tilde{Y} \in \Gamma(T\bar{M})$$

s.t $\tilde{X}|_M = X$, $\tilde{Y}|_M = Y$

If $\bar{D}$ denotes the Levi Civita on $\bar{M}$

$$D_X Y := \left(\bar{D}_{\tilde{X}} \tilde{Y}\right)^T$$

orthogonal projection a vector in $(T\bar{M}_p, g_p)$ onto TM_p:

$$T\bar{M}_p = TM_p \oplus TM_p^\perp$$

$\tilde{X}, \tilde{Y}$ are extentensions of of the v.f $X, Y \in \Gamma(TM)$

<u>exercise</u> show $D_X Y$ defines a connection on TM the is both compatible with g and torsion free.

$\longrightarrow$ D is the Levy-Civita!

e.g. Check compatibility:

tangent

$$X g(Y,Z) = \tilde{X} \bar{g}(\tilde{Y}, \tilde{Z})\big|_M = \bar{g}\left(\left(\bar{D}_{\tilde{X}} \tilde{Y}\right)^T, \tilde{Z}\right) + \bar{g}\left(\tilde{Y}, \left(\bar{D}_{\tilde{X}} \tilde{Z}\right)^T\right)$$

Def'n 2.17 $\mathbb{R}$-bilinear map $h : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM^\perp)$

$$h(X,Y) = (\bar{D}_{\tilde{X}} \tilde{Y})^\perp = \bar{D}_{\tilde{X}} \tilde{Y} - D_{\tilde{X}} Y$$

Observation: $h(X,Y) = h(Y,X)$. Indeed:

$$h(Y,X) = (\bar{D}_{\tilde{Y}} \tilde{X})^\perp = (\bar{D}_{\tilde{X}} \tilde{Y} + \underbrace{[\tilde{X}, \tilde{Y}]}_{\text{tangent}})^\perp = (\bar{D}_{\tilde{X}} \tilde{Y})^\perp = h(X,Y)$$

For $N \in \Gamma(TM^\perp)$, $\bar{g}(N,N) = 1$ (unit normal v.f. of M)

Define $h_N(X,Y) = \bar{g}(N, h(X,Y))$ is $C^\infty(M)$ homog.

i.e, it is $(0,2)$ tensor field

Associated $(1,1)$ tensor field S_N $\quad g(S_N(X), Y) := h_N(X,Y)$ $\forall X, Y \in \Gamma(TM)$

Lemma 2.18 $\quad S_N(X) = -(\bar{D}_X N)^T \underset{\text{ONLY}}{=} -\bar{D}_X N$

IF codimension
$\bar{m} - m = 1$

<u>proof</u>

$$g(S_N(X), Y) := \bar{g}(N, \overbrace{h(X,Y)}^{\bar{D}_X Y})$$

$$= \cancel{X \underbrace{\bar{g}(X,N)}_{0}} - \bar{g}(\bar{D}_X N, Y), \quad \forall Y \in \Gamma(TM).$$

$\bar{D}_X$ + compatibility

when $\bar{m} - m = 1$, $\quad g(N,N) \equiv 1 \implies 2g(\bar{D}_X N, N) \equiv 0$

$$\implies \bar{D}_X N \perp N \implies \bar{D}_X N = (\bar{D}_X N)^T$$

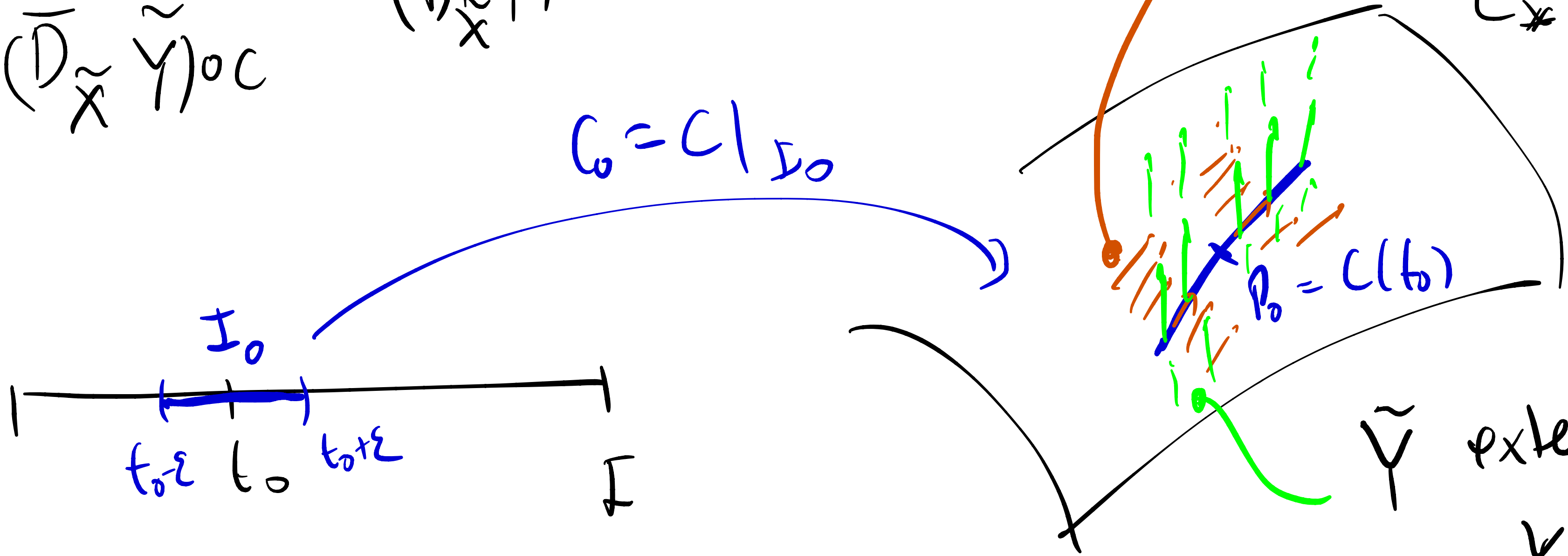
2^{n} fund. form $\longleftrightarrow$ extrinsic acceleration / curvature

Lemma 2.20 $\quad c : I \longrightarrow M \subset \bar{M}$ regular curve

$$Y \in \Gamma(c^*TM) \subset \Gamma(c^*T\bar{M}) \qquad (\text{recall } Y(t) \in TM_{c(t)})$$

$$\underbrace{\frac{\bar{D}}{dt} Y}_{(\bar{D}_{\tilde{X}} \tilde{Y}) \circ c} = \underbrace{\frac{D}{dt} Y}_{(D_{\tilde{X}} \tilde{Y}) \circ c} + \underbrace{h(\dot{c}, Y)}_{h(\tilde{X}, Y) \circ c}$$

$\tilde{X}$ extension of $c_* \frac{d}{dt} \circ c_0^{-1}$



$\tilde{Y}$ extension of $Y \circ c_0^{-1}$

<u>Example</u> $M = S^m$, $\bar{M} = \mathbb{R}^{m+1}$, $p \in S^m$ $v \in TS^m_p$ with $|v| = 1$

Goal: compute $h(v,v)$

Up to rotation (in $SO(m+1)$), $p = (1,0,0,\dots)$, $v = (0,1,0,\dots)$

The great circular arc (i.e geodesic) with $c(0) = p$, $\dot{c}(0) = v$

$$c(t) = (\cos t, \sin t, 0, \dots 0)$$

(lemma 7.20)

$$\frac{\bar{D}}{dt}\dot{c} = \ddot{c} = \cancel{\frac{D}{dt}\dot{c}}^{\,0} + h(v,v) \quad (\text{at } t=0)$$

$\ddot{c} = N$

inwards unit normal vector

$$\Rightarrow \boxed{h_N(v,v) = 1}$$

$$\Leftrightarrow \quad h(v,v) = N|v|^2 \quad \forall p \in M, \ \forall v \in TM_p$$

Thm 2.19 (Gauss eqns) $M, \bar{M}$ as above, $R, \bar{R}$ resp. Riem. tensors

$$R(V,W,X,Y) = \bar{R}(V,W,X,Y) + \bar{g}(h(V,X),h(W,Y)) - \bar{g}(h(V,Y),h(W,X))$$

$\forall\, V,W,X,Y \in \Gamma(TM)$

---

proof $R(V,W,X,Y) = g(V, D_X D_Y W - D_Y D_X W) = \dots$

$$D_Y W = (\bar{D}_Y W)^T = \bar{D}_Y W - \underbrace{h(W,Y)}_{(\bar{D}_Y W)^\perp}$$

$$D_X D_Y W = (\bar{D}_X \bar{D}_Y W)^T - (\bar{D}_X(h(W,Y)))^T$$

$$\ldots = \bar{g}\left(V, \left(\left(\bar{D}_X \bar{D}_Y - \bar{D}_Y \bar{D}_X\right) W\right)^{T}\right)$$

$$- \bar{g}\left(V, \left(\bar{D}_X h(W,Y)\right)^{T}\right) + \bar{g}\left(V, \left(\bar{D}_Y h(W,X)\right)^{T}\right)$$

$$= \bar{g}\left(V, \bar{R}(X,Y)W\right) - \quad + \quad .$$

Observe: $V \in \Gamma(TM)$, $h(W,Y) \in \Gamma(TM^{\perp})$

$$\bar{D}_X\left(\bar{g}(V, h(W,Y)) = 0\right)$$

$$\bar{g}\left((\bar{D}_X V)^{T}, \underbrace{h(W,Y)}_{TM^{\perp}}\right) = -\bar{g}\left(V, \bar{D}_X h(W,Y)\right)$$

$$\parallel$$

$$\bar{g}\left(h(X,V), h(W,Y)\right)$$

(analogously with other term)

$\blacksquare$

<u>Application</u> $(V, W, X, Y) \leftarrow (e_1, e_2, e_1, e_2)$ e_1, e_2

"replace by" ONB of

T_pM

$$R(e_1, e_2, e_1, e_2) = \bar{R}(e_1, e_2, e_1, e_2) + \bar{g}(h(e_1, e_1), h(e_2, e_2)) - \bar{g}(h(e_1, e_2), h(e_1, e_2))$$

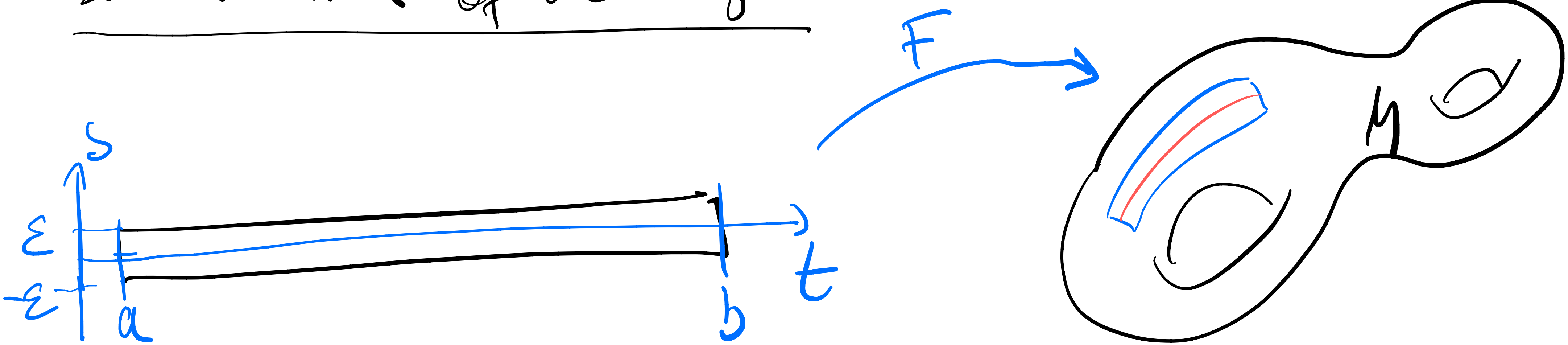
Relation between sec curv.

If $\bar{M} = \mathbb{R}^3$ $M \subset \mathbb{R}^3$ 2-dim surf is

<u>Th$^{\underline{a}}$ Egregium</u>

<u>Exercise</u> Show sec $\mathbb{S}^m$ is ct $= 1$, $r\,\mathbb{S}^m$ is ct $= \frac{1}{r^2}$

2nd variation of arc length



F immersion from $(-\varepsilon,\varepsilon)\times[a,b] \longrightarrow M$

$$\gamma_s(t) = F(s,t)$$

<u>Goal</u> $\quad \dfrac{d}{ds}\Big|_{s=0} L(\gamma_s) = \,?$ $\qquad$ 1st variation

$$\frac{d^2}{ds^2}\Big|_{s=0} L(\gamma_s) = ?$$

2nd variation

<u>Thm 3.1</u> Define $V = F_* \frac{\partial}{\partial s}$, $T = F_* \frac{\partial}{\partial t}$

$V_s(t) := V(s,t)$, then if γ_0 is a unit speed <u>geodesic</u>

$$\frac{d^2}{ds^2} L(\gamma_s) = \int_a^b \left(|(V_0')^\perp|^2 - R(V_0, \gamma_0', V_0, \gamma_0') \right) dt$$

$$+ \left\langle \left(\frac{D}{\partial s} V \right)\Big|_{s=0}, \gamma_0' \right\rangle \Big|_a^b$$

where $'$ means $\frac{D}{dt}$ or $\frac{d}{dt}$

and $\perp$ means orthogonal proj onto $(\gamma_0'(t))^\perp$

$\gamma_0'(t)$ is velocity vector of γ_0 at $\gamma_0(t) \in M$

$w \in TM_{\gamma(t)}$ we can do the othogonal decomp.

$$w = w^T + w^\perp$$

$$w^T = \frac{\langle w, \gamma_0'(t) \rangle}{|\gamma_0'(t)|} \frac{\gamma_0'(t)}{|\gamma_0'(t)|}$$

$$L(\gamma_s) = \int_a^b |\gamma_s'(t)|\,dt$$

$$\frac{d}{ds}|\gamma_s'(t)| = \frac{d}{ds}\sqrt{\langle \gamma_s'(t), \gamma_s'(t)\rangle}$$

$$\overset{(*)}{=} \frac{1}{2|\gamma_s'(t)|}\, 2\left\langle \frac{D}{\partial s}\gamma_s'(t), \gamma_s'(t)\right\rangle$$

$$= \frac{1}{|\gamma_s'(t)|}\left\langle \frac{D}{\partial s}T, T\right\rangle$$

$T = F_* \frac{\partial}{\partial t}$

vector field along F

So, if $|\gamma_0'(t)| \equiv \lambda$

$$\begin{cases} \tilde{X} = F_* X|_{\mathrm{im}\, F} \circ F^{-1} \\ \tilde{T} = T \circ F^{-1} \end{cases}$$

If X is vector field in $(-\varepsilon, \varepsilon) \times [a,b]$

$$D_X T = D_{\tilde{X}} \tilde{T} \circ F$$

$$\frac{d}{ds}\Big|_{s=0} L(\gamma_s) = \int_a^b \frac{d}{ds}\Big|_{s=0} |\gamma_s'(t)|\, dt$$

$$= \frac{1}{\lambda}\int_a^b \left\langle \frac{D}{\partial s} T, T \right\rangle\Big|_{s=0} dt$$

D is torsion free

$[V,T] \equiv 0$

$$= \frac{1}{\lambda}\int_a^b \left\langle \frac{D}{\partial t} V, T \right\rangle\Big|_{s=0} dt$$

$$= \frac{1}{\lambda}\int_a^b \frac{d}{dt}\langle V, T\rangle\Big|_{s=0} - \left\langle V, \frac{D}{\partial t} T\right\rangle\Big|_{s=0}$$

$$= \frac{1}{\lambda}\left(\langle V_0, \gamma_0' \rangle\Big|_a^b - \int_a^b \langle V_0, \gamma_0'' \rangle\, dt \right)$$

$$\frac{d^2}{ds^2}\bigg|_{s=0} L(\gamma_s) = \frac{d}{ds}\bigg|_{s=0}\left(\frac{d}{ds} L(\gamma_s)\right)$$

$$\overset{(*)}{=} \frac{d}{ds}\bigg|_{s=0} \int_a^b \frac{1}{|T|}\left\langle \frac{D}{\partial s} T, T\right\rangle dt$$

$$\frac{\partial}{\partial s}\bigg|_{s=0} \frac{1}{|T|}\left\langle \frac{D}{\partial s} T, T\right\rangle = V\left(\frac{1}{|T|}\left\langle \frac{D}{\partial s} T, T\right\rangle\right)\bigg|_{s=0}$$

$$= \frac{1}{|T|}\left(\underbrace{\left\langle \frac{D}{\partial s}\frac{D}{\partial s} T, T\right\rangle}_{\text{1st term}} + \left\langle \frac{D}{\partial s} T, \frac{D}{\partial s} T\right\rangle\right) \quad \frac{D}{dt} V$$

$$- \frac{1}{|T|^{3/2}}\left\langle \frac{D}{\partial s} T, T\right\rangle^2$$

$$\frac{1}{|T|} = \langle T, T\rangle^{-1/2}$$

1st term

$$\left\langle \overbrace{\frac{D}{\partial s}\frac{D}{\partial s} T}^{(D_{\tilde V} D_{\tilde V}\tilde T)\circ F}, T \right\rangle \overset{D \text{ torsion free}}{=} \left\langle \overbrace{\frac{D}{\partial s}\frac{D}{\partial t} V}^{(D_{\tilde V} D_{\tilde T}\tilde V)\circ F}, T \right\rangle$$

$$= \left\langle \overbrace{\frac{D}{\partial t}\frac{D}{\partial s} V}^{(D_{\tilde T} D_{\tilde V}\tilde V)\circ F}, T \right\rangle - \left\langle R(T,V)V, T \right\rangle$$

$$\left\langle \frac{D}{\partial t}\frac{D}{\partial s} V, T \right\rangle \overset{\text{Comp.}}{=} T\left\langle \frac{D}{\partial s} V, T \right\rangle - \left\langle \frac{D}{\partial s} V, \underbrace{\frac{D}{\partial t} T}_{} \right\rangle$$

when evaluating at $s=0$

$$\frac{D}{dt}\gamma_0' = 0$$

γ_0 geodesic $\Longleftarrow$

Putting this together, we get:

$$\frac{d^2}{ds^2}\Big|_{s=0} L(\gamma_s) = \int_a^b \frac{1}{|T|}\Big(\overbrace{|\tfrac{D}{\partial t}V|^2 - \big\langle \tfrac{D}{\partial t}V, \tfrac{T}{|T|}\big\rangle^2}^{|(\frac{D}{dt}V)^{\perp}|^2}$$

$$- R(V,T,V,T) + T\big\langle \tfrac{D}{\partial s}T, T\big\rangle\Big)\Big|_{s=0} dt$$

$$= \int_a^b \Big(|(\tfrac{D}{dt}V_0)^{\perp}|^2 - R(V_0,c',V_0,c')\Big)\,dt$$

$$+ \int_a^b \frac{d}{dt}\big\langle \tfrac{D}{\partial s}T, T\big\rangle\Big|_{s=0} dt$$

Observation We have much freedom to choose variations F as in the 2^{nd} var. thm. (around a given geodesic $t \longmapsto c(t)$

$$c : [a,b] \longrightarrow (M,g) \qquad \text{geodesic} \qquad |c'| = 1$$

How to construct a smooth immersions

$$F : (-\varepsilon, \varepsilon) \times [a,b] \longrightarrow M \qquad \text{s.t.}$$

$$F|_{s=0}(t) = c(t)$$

Given a normal vector field along c:

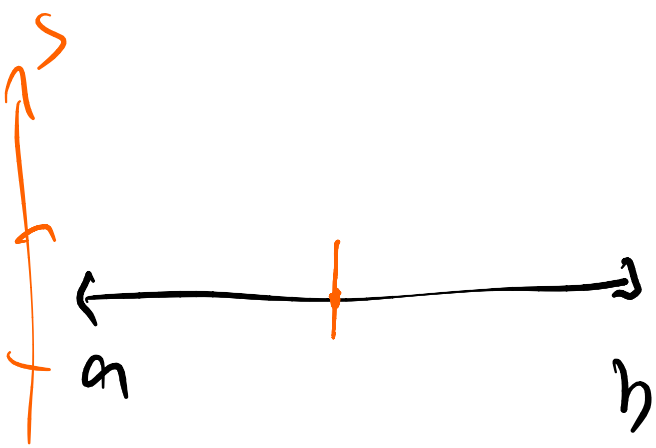
$N \in \Gamma(c^* TM)$ st $N(t) \perp c'(t)$, $\|N(t)\| > 0$

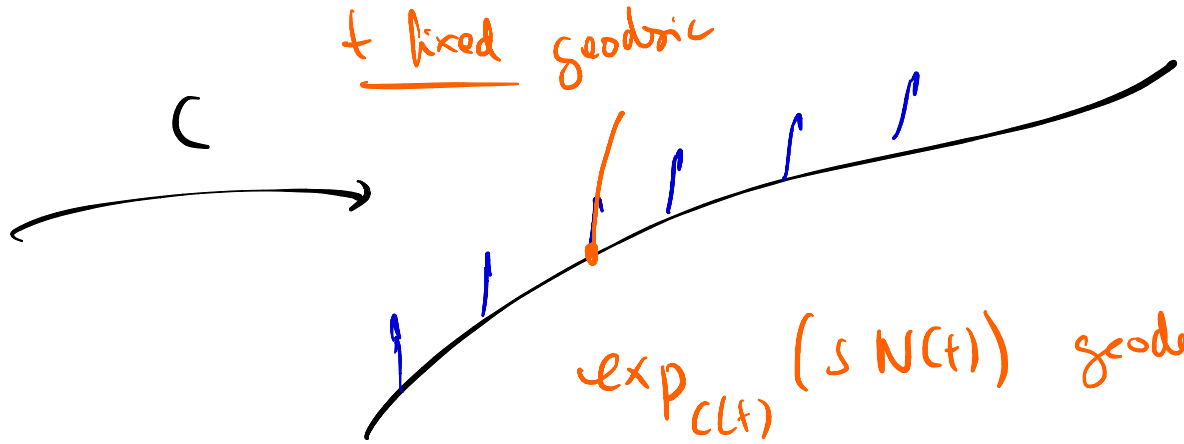
If $M = \mathbb{R}^m$ $F(s,t) = c(t) + s N(t)$

In general $F(s,t) = \exp_{c(t)}(s N(t))$

is a smooth variation with $V_0(t) = N(t)$

$$\frac{D}{\partial s} V \Big|_{s=0} = 0$$





$\exp_{c(t)}(s\,N(t))$ geodesic $\bar{c}$

$$\bar{c}(0) = c(t), \quad \bar{c}'(0) = N(t)$$

$$\frac{D}{ds}\bar{c}'(s) = 0 \implies \frac{D}{\partial s} V \equiv 0$$

We say that a Riem. mfld is closed if it is compact (without bdry!)

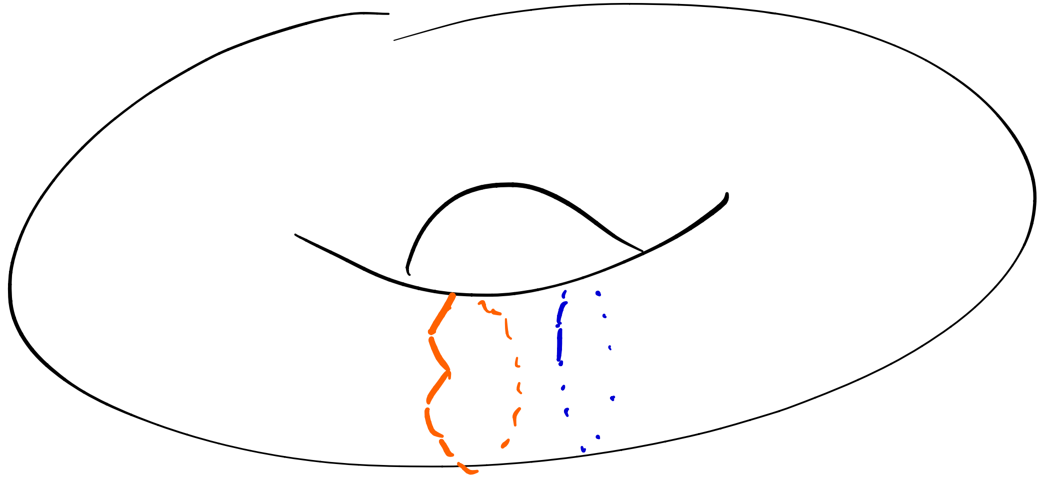
Thm 3.2 (Synge 1936) M is a closed Riem. of even dimension and positive sec. curvatures

M is orientable $\Rightarrow$ M simply connected

---

proof Suppose M as in statement, orientable but **NOT** simply connected $\Rightarrow$ $\exists$ closed curve $\alpha : [0,1] \to M$ not homotopic to a constant curve

Then $\exists$ minimizing geodesic in the same free homotopy class as α. Call it $c : [0, \ell] \to M$ unit speed closed geodesic "rubber + oil"



$\mathbb{R}^3$

$\perp$ to $c'(0) = c'(\ell)$

$p = c(0) = c(\ell)$

Consider the linear space $H := TM_p^{\perp}$

Given $v \in TM_p^{\perp}$ let $V(t)$ be the parallel transport along c of v (namely, $\frac{D}{dt} V \equiv 0$ $\quad V(0) = v$)

Notice that parallel transport preserves scalar product

$V(t), W(t)$ are both parallel along c

$$\langle V(t), W(t)\rangle \equiv ctt$$

$$\frac{d}{dt}\langle V(t), W(t)\rangle = \langle \overbrace{\frac{D}{dt} V(t)}^{0}, W(t)\rangle + \langle V(t), \overbrace{\frac{D}{dt} W(t)}^{0}\rangle$$

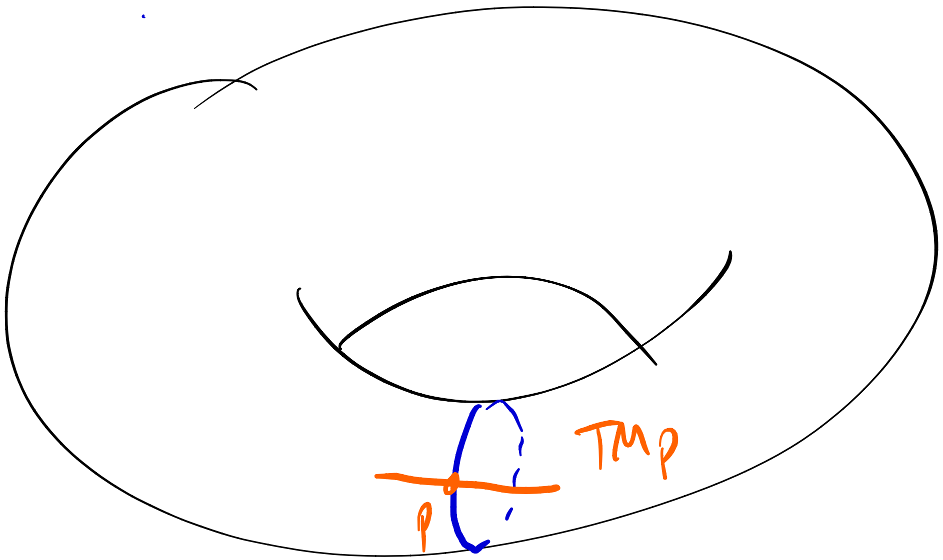
$$= 0$$

$\Longrightarrow$ $V(t)$ is $\perp$ to $c'(t)$ $\forall t$ $\quad (W \equiv c')$

I can define <u>linear</u> map $P : H \longrightarrow H$

$$v \longmapsto V(\ell)$$

which gives an isometry of (H, g_p)



Since M is orientable, P is a positive isometry of an odd dimensional linear space (H, g_p)

$\exists$ eigenvector v_0 of P st $Pv_0 = v_0$

Use 2^{nd} variation formula with $F(s,t) = \exp_{c(t)}(sN(t))$

$N(t) = V_0(t)$ $\qquad$ $\gamma_s(t) = F(s,t)$

By Thm 3.1, and minimality of c

$$0 \leq \frac{d^2}{ds^2}\Big|_{s=0} L(\gamma_s) = - \int_a^b \underbrace{R(N(t), c'(t), N(t), c'(t))}_{\substack{\text{sec in plane}\\ \text{generated by}\\ N(t),\, c'(t) \,\subset\, TM_{c(t)}}} dt$$

if $|N| \equiv 1$

$$< 0$$

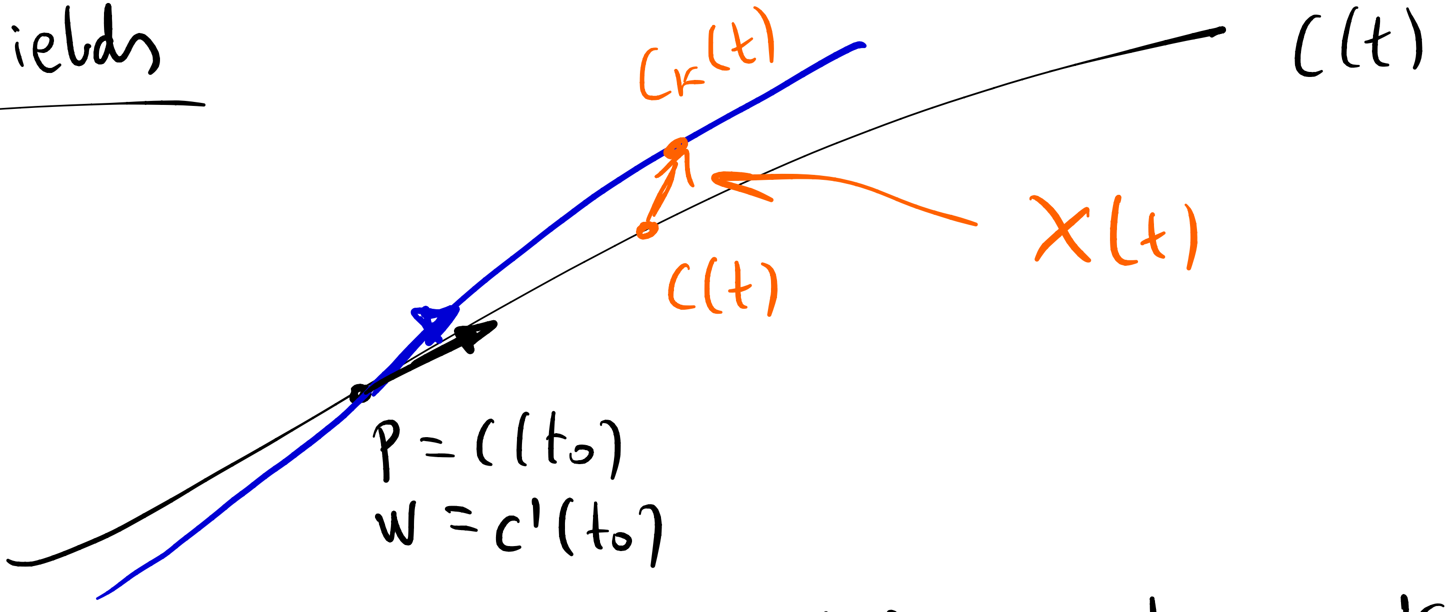
$\blacksquare$

<u>Rem.</u> In odd dim. $2k+1$

$$\mathbb{R}P^{2k+1} = (S^{2k+1}, g)/\sim \qquad x \sim -x$$

shows assumption on even dim. is necessary.

Jacobi Fields



Consider $c_k(t)$ geodesics $k \geq 1$

$$c_k(t_0) = p$$

$$c_k'(t_0) = w_k \to w$$

If $r_k := \|w_k - w\|_{\theta_p}$ $\quad X(t)\, f = \lim_k \dfrac{f(c_k(t)) - f(c(t))}{r_k}$

$$\forall f \in C^\infty(M)$$

Def'n A vector field Y along a geodesic $c : I \to M$ is called *Jacobi field* if it satisfies

$$\frac{D}{dt}\frac{D}{dt} Y + R(Y, c')c' = 0$$

(In brief $Y'' + R(Y, c')c' = 0$

Lemma 3.5 The Jacobi field along a given geodesic $c : I \to M^m$ form a $2m$-dim vector space. For $t_0 \in I$, $v, w \in TM_{c(t_0)}$ there is a unique Jacobi field Y along c with $Y(t_0) = v$ and $Y'(t_0) = w$

Useful trick (see proof of Lem 3.5 in notes)

$c : [0, \ell] \longrightarrow M$ geodesic

$e_i \in TM_{c(0)}$ ONB of $(TM_{c(0)}, g)$

$E_i \in \Gamma(c^* TM)$ parallel transport $\left\{ \begin{array}{l} \frac{D}{dt} E_i \equiv 0 \\ E_i(0) = e_i \end{array} \right.$

$E_i(t)$ is ONB of $TM_{c(t)}$

Express Jacobi fields $Y(t) = Y^i E_i$

$Y^i : [0, \ell] \longrightarrow \mathbb{R}$ C^∞ functions

$Y'' + R(Y, c')c' = 0 \iff \ddot{Y}^i E_i + R(Y^i E_i, c')c'$

$$v \in TM_{c(t)} \longmapsto R(v, c')c' \in TM_{c(t)}$$

$$R(E_i, c')c' = C_i^j E_j \,, \qquad C_i^j = C_i^j(t)$$

$$\ddot{Y}^j E_j + C_i^j Y^i E_j \equiv 0$$

$$\iff \ddot{Y}^j + C_i^j Y^i = 0 \qquad \begin{pmatrix} Y^1 \\ \vdots \\ Y^n \end{pmatrix} : [0, \ell] \to \mathbb{R}^n$$

<u>Prop 3.6</u> $c : [0, \ell] \to M$ geodesic $|c'| = 1$.

The following two are equivalent

(1) $\exists\, \varepsilon > 0$, $F = F(s,t)$ immersion of $(-\varepsilon, \varepsilon) \times [0, T]$ in M s.t $\gamma_s(t) := F(s,t)$ is geodesic $\forall s$

and $Y(t) = V_0(t) = \left(F_* \frac{\partial}{\partial s}\right)(0, t)$

(2) $Y(t)$ is a Jacobi field along c with $|Y| \neq 0$

(1) $\Rightarrow$ (2) similarly as in Thm 3.1

$$V = F_* \frac{\partial}{\partial s}, \quad T = F_* \frac{\partial}{\partial t}$$

$F(s,\cdot)$ is geodesic $\iff$ $\frac{D}{\partial t} T \equiv 0$ $\quad \frac{D}{\partial t}\frac{D}{\partial t} V$

$$0 = \underbrace{\frac{D}{\partial s}\frac{D}{\partial t} T}_{(D_{\tilde V} D_{\tilde T} \tilde T)\circ F} = \underbrace{\boxed{\frac{D}{\partial t}\frac{D}{\partial s} T}}_{\substack{(D_{\tilde T} D_{\tilde V} \tilde T)\circ F \\ \| \\ (D_{\tilde T} D_{\tilde T} \tilde V)\circ F}} + \underbrace{R(V,T)\,T}_{R(\tilde V, \tilde T)\tilde T \circ F} \qquad (*)$$

<u>Recall:</u>

$\tilde V$ extension of $V \circ F^{-1}$

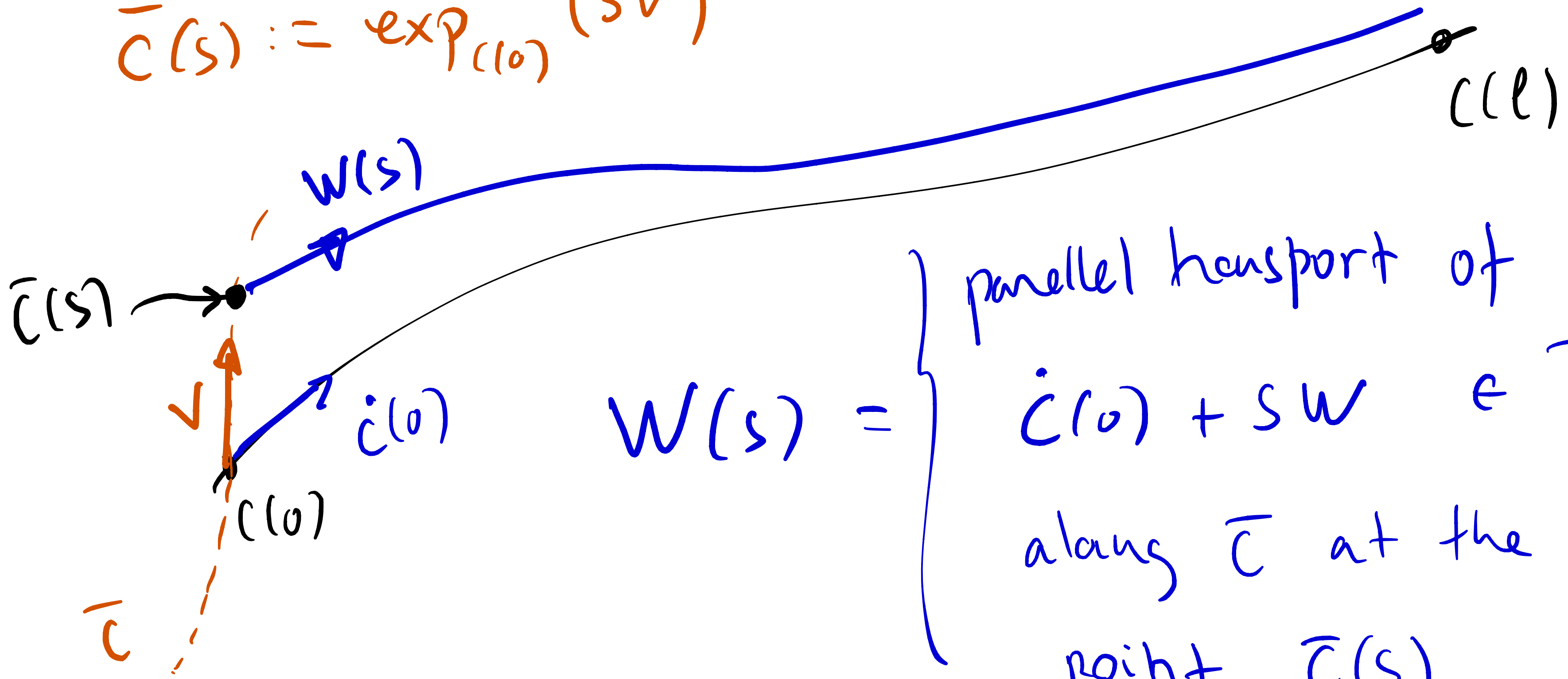
$\tilde T$ extension of $T \circ F^{-1}$

evaluate $(*)$ at $s=0$ $\Rightarrow$ V_0 is a Jacobi field

$(2) \Rightarrow (1)$ Let $Y = Y(t)$ be a J.F. along c, $|Y| \neq 0$,

Given $v, w \in TM_{c(0)}$ with $v, \dot{c}(0)$ l.i. we have

$\bar{c}(s) := \exp_{c(0)}(sv)$ $\quad s \in (-\varepsilon, \varepsilon)$



$$W(s) = \left\{ \begin{array}{l} \text{parallel transport of} \\ \dot{c}(0) + sw \in TM_{c(0)} \\ \text{along } \bar{c} \text{ at the} \\ \text{point } \bar{c}(s) \end{array} \right.$$

Define $F(s,t) = \exp_{\bar{c}(s)}(w(s)\,t)$

Notice F is smooth (if $\varepsilon > 0$ is small)

$F(0,s) \equiv \bar{c}(s)$

$V_0(0) = F_* \frac{\partial}{\partial s}\Big|_{t=0} = \bar{c}'(s) \Rightarrow F_* \frac{\partial}{\partial s}\Big|_{s=t=0} = v$

$F_* \frac{\partial}{\partial t}\Big|_{t=0} = d\left(\exp_{\bar{c}(s)}\right)_0 (W(s)) \overset{(☺)}{=} W(s) \sim c'(0)$

$\forall\, \varepsilon > 0$ small $\exists\, t_\varepsilon$ s.t. $F|_{(-\varepsilon,\varepsilon)\times[0,t_\varepsilon]}$ is immersion

Compute

$$\frac{D}{\partial t} F_* \frac{\partial}{\partial s}\Big|_{(0,0)} = \frac{D}{\partial s}\Big|_{s=0} F_* \frac{\partial}{\partial t}(\cdot, 0) = \frac{D}{\partial s} W(s) = w$$

$$\| $$

$$V_0'(0) = w$$

$V_0 = F_* \frac{\partial}{\partial s}\Big|_{s=0}$ by 1st part of proof is a

Jacobi field along $c|_{[0, t_\varepsilon)}$

Chose $v = Y(0)$, $w = Y'(0)$ $\Rightarrow$

$\boxed{V_0 = Y}$ $t \in [0, t_\varepsilon]$

By assumption $|Y|>0$ in $[0,\ell]$ for $\varepsilon>0$ sufficiently small $t_\varepsilon=\ell$.

<u>Remark 1</u> $p \in M$ $\quad F(s,t) = \exp_p\big(t(v+sw)\big)$

$$Y(t) = F_* \frac{\partial}{\partial s}\Big|_{s=0} \text{ is J.F. along } \exp_p(tv)$$

$$Y(t) = d(\exp_p)_{tv}(tw)$$

$$Y(0) = 0$$

$$Y'(0) = d(\exp_p)_0(w) = w$$

Remark 2 Y JF along c (geodesic) $|c'| \equiv 1$

$$\langle Y, c'\rangle'' = \left(\langle Y', c'\rangle + \langle Y, \overset{0}{\cancel{c''}}\rangle\right)'$$

$$= \langle Y'', c'\rangle = -\langle R(Y,c')c', c'\rangle \equiv 0$$

$$\Rightarrow \quad \langle Y, c'\rangle = at + b$$

$$Y^T = \langle Y, c'\rangle c' \quad \Rightarrow \quad (Y^T)'' = \left(\langle Y, c'\rangle c'\right)'' \equiv 0$$

$$R(\underbrace{Y^T}_{\text{parallel to } c'}, c')c' \equiv 0$$

$$\Rightarrow \quad Y^\perp = Y - Y^T \text{ is J.F.}$$

"the interesting one!"

<u>Rem. 3</u> M <u>space form</u> (sec. curv $\equiv k$)

$$R(X, c')c' = kX$$

$c : [0, \ell] \longrightarrow M$ is geodesic, consider E parallel v.f along c s.t $E(t) \perp c'(t)$

$$\boxed{Y = fE} \quad \big(f : [0,\ell] \longrightarrow \mathbb{R} \big)$$

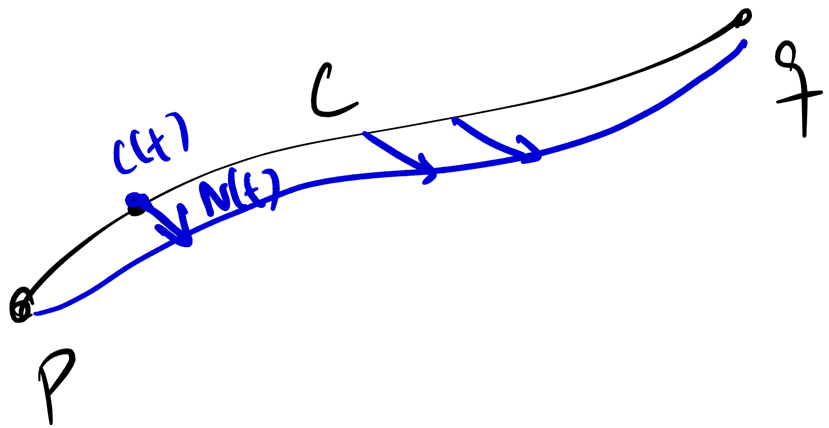
$$Y'' = (f'E)' = \underbrace{f''E}_{} \overset{\text{J.F. eq'n}}{=} -R(Y, c', c') = -kfY = -k\underbrace{fE}_{}$$

J.F. eq'n reads $\quad f'' + kf = 0$

Rem. 4 If M complete connected

$\forall p, q \in M$, $\exists\, c : [0, \ell] \to M$ length minimizing joining p, q

$|c'| \equiv 1$



For all $N = N(t) \in \Gamma(c^* TM)$ s.t $N(t) \perp c'(t)$, $N \neq 0$

consider
$$F(s,t) = \exp_{c(t)}(sN(t))$$

$$\gamma_s = F(s, \cdot)$$

$$\frac{d}{ds}\Big|_{s=0} L(\gamma_s) = \cancel{\left[\langle N, c' \rangle\right]_0^\ell} - \int_0^\ell \cancel{\langle N, c'' \rangle}\, dt$$

$$= 0$$

$$\frac{d^2}{ds^2}\Big|_{s=0} L(\gamma_s) = \int_0^\ell |(N')^\perp|^2 - R(N, c', N, c')\, dt$$

$$\cancel{+ \left[\frac{D}{\partial s} F_* \frac{\partial}{\partial s}\Big|_{s=0}, c'\right]_0^\ell}$$

By approx. one can take $N(0) = N(\ell) = 0$

proper variation

$$\gamma_s(0) = c(0) = p \quad \Big| \quad \gamma_s(\ell) = c(\ell) = q$$

By minimality of c we deduce

$$0 \leq \int_0^{\ell} |\underbrace{(N')^{\perp}}_{N'}|^2 - R(N, c', N, c')\, dt$$

For every N as above

Observation: if $N \perp c'$ (c geodesic) $\Rightarrow N' \perp c'$

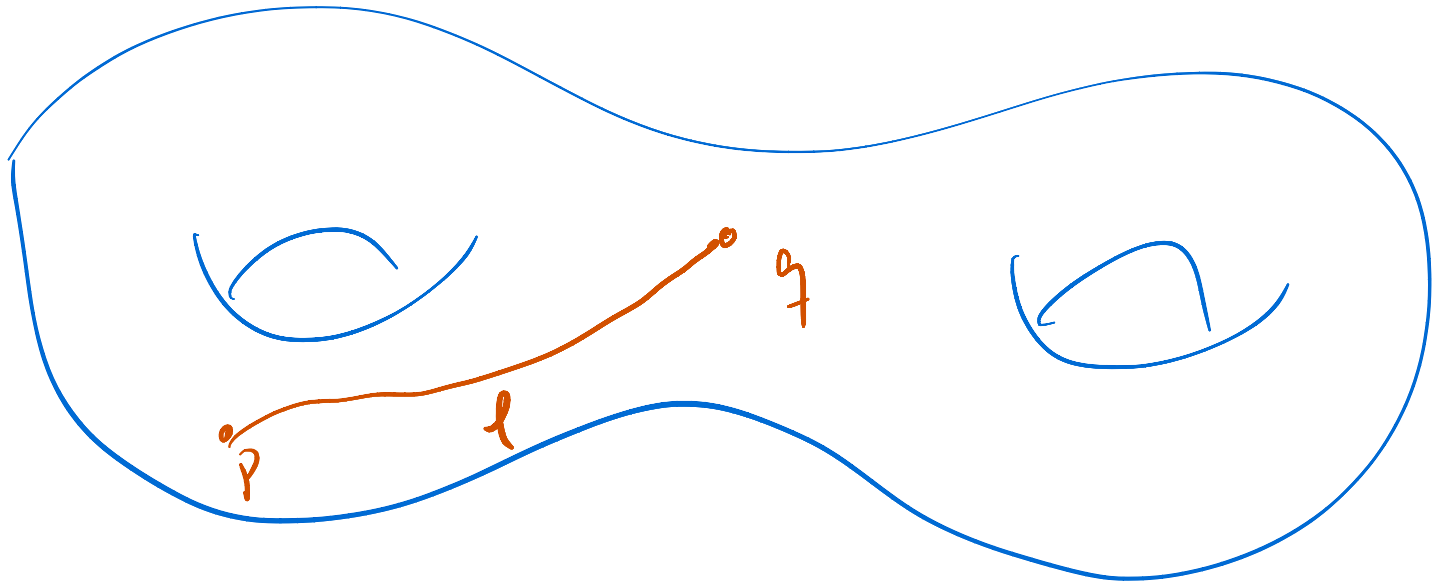
$$\langle N, c' \rangle \equiv 0 \Rightarrow \langle N', c' \rangle = -\langle N, c'' \rangle \equiv 0$$

Thm 3.9 (Myers, 1941) Let (M,g) complete Rien. mfld with $\boxed{Ric(v,v) \geq (m-1)k}$ $(*)$ for some $k>0$

Then

$$diam(M) := \sup\{d(p,q) \mid p,q \in M\} \leq \frac{\pi}{\sqrt{k}}$$

proof



Fix $p,q \in M$, M is complete $\exists c$ unit speed geodesic with $c(0)=p$, $c(\ell)=q$ s.t. $\ell = d(p,q)$

<u>Goal</u> bound ℓ

$$0 \le \int_0^\ell |N'|^2 - R(N,c',N,c')\,dt \qquad \forall N \perp c' \quad (**)$$

st $N(0)=N(\ell)=0$

Choose $E_1, \dots, E_m$ parellel ONB along c st. $E_m = c'$

test eq'n $(**)$ with $N = fE_i$ $\quad i = 1, \dots, m-1$, $f = f(t)$

$f(0) = f(\ell) = 0$

$$0 \le \int_0^\ell (f')^2 \underbrace{|E_i|^2}_{1} - \underbrace{R(fE_i, c', fE_i, c')}_{f^2 R(E_i, c', E_i, c')}\,dt$$

$$\sum_{i=1}^{m-1} \qquad 0 \leq \int_0^\ell (m-1)(f')^2 - \underbrace{\mathrm{Ric}(c',c')}_{\substack{\vee\mathrm{I} \\ (m-1)K}} f^2 \, dt$$

$$\leq (m-1) \int_0^\ell \left((f')^2 - K f^2 \right) dt \qquad \forall f : [0,\ell] \to \mathbb{R} \text{ s.t. } f(0) = f(\ell) = 0$$

$$\Longleftrightarrow \qquad K \leq \frac{\int_0^\ell (f')^2}{\int_0^\ell f^2}$$

Best possible f : $\qquad f(t) = \sin\left(\frac{\pi}{\ell} t\right)$

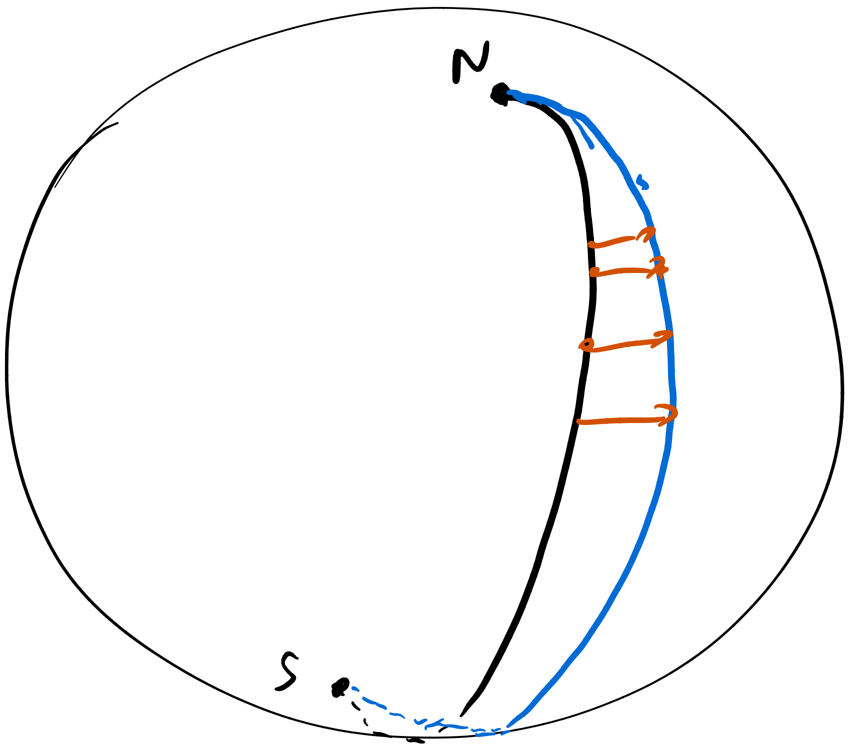
for this f $\quad k \leq \dfrac{\int_0^l \cos^2\left(\frac{\pi}{l} t\right)\left(\frac{\pi}{l}\right)^2 dt}{\int_0^l \sin^2\left(\frac{\pi}{l} t\right) dt} = \left(\frac{\pi}{l}\right)^2$

$$\Longrightarrow \quad l = d(p,q) \leq \frac{\pi}{\sqrt{k}}$$

sup over p, q to get bound on $\mathrm{diam}(M)$ $\blacksquare$

<u>Exercise</u> Check that for $S_r \Rightarrow k = \frac{1}{r^2}$

"all" the inequalities are equalities, when $p =$ North $q =$ South

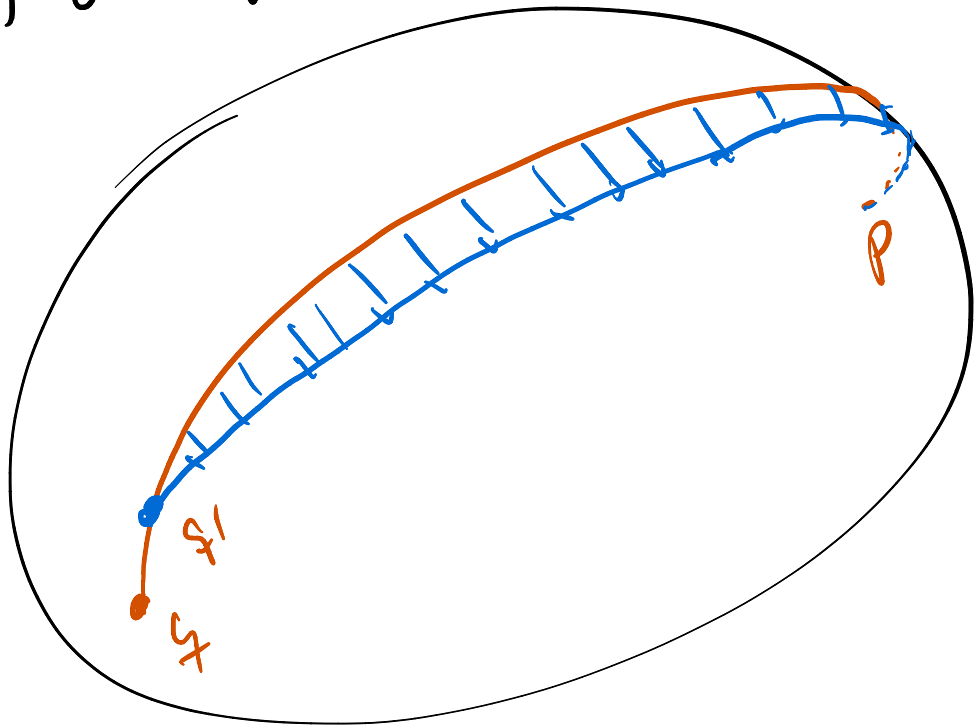


<u>Conjugate pts</u> $\quad c : [a,b] \longrightarrow M$ from p to q

we say p is conjugate to q along c if $\exists$ a non-trivial (i.e $\not\equiv 0$) Jacobi field Y s.t. $Y(a) = 0$

$Y(b) = 0$

We will show that a geodesic is <u>never</u> minimizing past a conjugate pt.



<u>Goal</u> Given geodesic $c : [0, \ell] \to M$. Suppose $\exists \ell' < \ell$ s.t. $c(\ell')$ is conj. to $c(0)$ (along c) then the $\exists N \perp c'$ with $\boxed{N(0) = N(\ell) = 0}$ s.t $F(s,t) = \exp_{c(t)} sN(t)$

gives negative 2^{nd} variation of length. (In part. $c|_{[0,\ell)}$ cannot be minimizing)

$$\gamma_s = F(s,t) \quad (\text{notice } \gamma_s(0) = p, \ \gamma_s(\ell) = q)$$

$$\frac{d^2}{ds^2}\Big|_{s=0} L(\gamma_s) = \overbrace{\int_0^\ell \langle N', N'\rangle - \underbrace{\langle N, R(N, c')c'\rangle}_{R(N,c',N,c')}}^{:= I_\ell(N,N)} < 0$$

Would like to find N "extremizer" of

$$Q(Z) := \frac{I_\ell(Z,Z)}{\int_0^\ell |Z|^2 dt} \qquad \Bigg| \qquad \lambda := \inf_{\substack{Z \perp c' \\ Z(0)=Z(\ell)=0}} Q(Z)$$

where $I_\ell(X,Y) := \int_0^\ell \langle X', Y' \rangle - R(X, c', Y, c')\, dt$

bilinear symmetric acting on $X, Y \in \Gamma(c^* TM)$

is the index form

(☆)

Notice $I_\ell(X,Y) = \int_0^\ell \langle X, Y' \rangle' - \langle X, \underbrace{Y'' + R(Y, c')c'} \rangle\, dt$

this equals 0
if Y is a Jacobi field!
(J.f.)

If Y is J.f.

$$I_\ell(X,Y) = \Big[\langle X, Y' \rangle \Big]_0^\ell$$

Minimizing Q ...

Step 1 find the ODE, assume the inf λ is attained by some $z \perp c'$, smooth with $z(0) = z(\ell) = 0$

$$Q(z) = \frac{J_\ell(z, z)}{\int_0^\ell |z|^2 \, dt}$$

$$\lambda = Q(z) \leq Q(z_\varepsilon) \quad ☺$$

for all $z_\varepsilon = z + \varepsilon X$

$X \perp c'$ with $X(0) = X(\ell) = 0$

For X fixed, we obtain the necessary condition for ☺

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} Q(z_\varepsilon) = 0$$

This is the good old principle of calc. var.

$$Q_\ell(z_\varepsilon) = \frac{I_\ell(z_\varepsilon, z_\varepsilon)}{\int_0^\ell \langle z_\varepsilon, z_\varepsilon\rangle dt}$$

$$= \frac{I_\ell(z,z) + 2\varepsilon I_\ell(z,X) + \varepsilon^2 I_\ell(X,X)}{\int_0^\ell \langle z,z\rangle + 2\varepsilon\langle z,X\rangle + \varepsilon^2\langle X,X\rangle}$$

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} Q(z_\varepsilon) = 2\frac{I_\ell(z,X)}{\int_0^\ell\langle z,z\rangle dt} - 2\frac{I_\ell(z,z)}{\left(\int_0^\ell\langle z,z\rangle dt\right)}\int_0^\ell\langle z,X\rangle dt$$

$$= \frac{2}{\int_0^\ell |z|^2 dt}\left(I_\ell(z,X) - \lambda\int_0^\ell\langle z,X\rangle dt\right)$$

Using $(\star)$, we obtain $\forall X \perp c'$ $\underbrace{X(0)=X(\ell)=0}$

$$0 = \cancel{\left[\langle z', X\rangle\right]_0^\ell} - \int_0^\ell \left(\langle z'' + R(z,c')c', X\rangle + \lambda\langle z, X\rangle \right) dt$$

$$\int_0^\ell \langle z'' + R(z,c')c' + \lambda z, X\rangle = 0$$

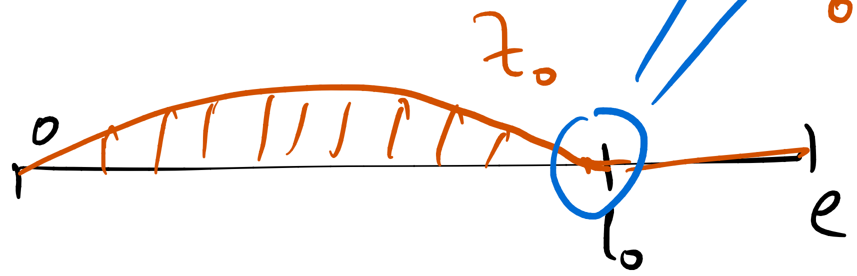
$(X \text{ is arbitrary}) \implies$ $\boxed{z'' + R(z,c')c' + \lambda z \equiv 0}$

$c : [0,\ell] \longrightarrow M$ geodesic, $l_0 \in (0,\ell)$

Exercise

1. $c(0)$ conjugate to $c(\ell_0)$ $\Longleftrightarrow$

$$\lambda_0 = \min_{\substack{z \perp c' \\ z(0)=z(\ell_0)=0}} Q_{\ell_0}(z) = 0 \qquad \arg\min\ z_0$$



corner

otherwise it would be $z_0'(\ell_0) = z_0(\ell_0) = 0$

2. So, if $\ell > \ell_0$

$$\lambda = \min_{\substack{z \perp c' \\ z(0)=z(\ell)=0}} Q_\ell < 0$$

Because

$$z(t) = \begin{cases} z_0(t) & \text{if } t \in [0, \ell_0] \\ 0 & \text{if } t \in [\ell_0, \ell] \end{cases}$$

already achieves $Q_\ell(z) = 0$!

Prop 3.17 (1st index lemma) $c : [0,\ell] \to M$ geodesic $|\dot c| \equiv 1$, s.t. $c(t)$ not conjugate to $c(0)$ (along $c|_{[0,t]}$) for all $t \in [0,\ell]$.

Let X be a piecewise smooth (continuous) v.f. along c and Y be a J.f. satisfying $Y(0) = X(0)$, $Y(\ell) = X(\ell)$.

Then $I_\ell(X,X) \geq I_\ell(Y,Y)$ ($= \Leftrightarrow X \equiv Y$)

$(X \mapsto X-Y)$ Y J.f.

$(X-Y)(0)=0$ use $(\star)$
$(X-Y)(\ell)=0$

proof $I(X,X) - I(Y,Y) = I(X-Y, X+Y)$

$$= I_\ell(X-Y, X-Y) + 2\, \underbrace{I(X-Y, Y)}_{= 0}$$

$\lambda > 0$

$$\geq \lambda \int_0^\ell |X-Y|^2$$

"Assumption 3.18" (for Rauch thm)

Suppose $M, \bar{M}$ two Riem. mflds $\dim(\bar{M}) \geq \dim(M) \geq 2$.

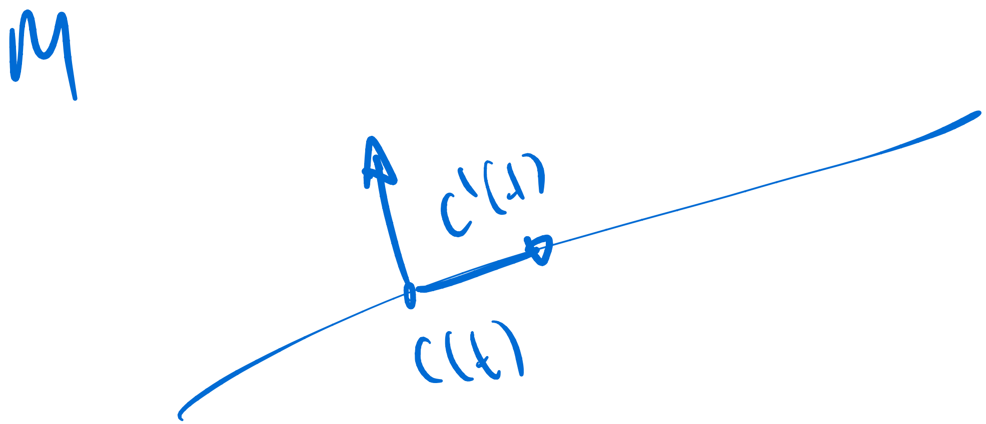
Assume that we have two unit speed geod.

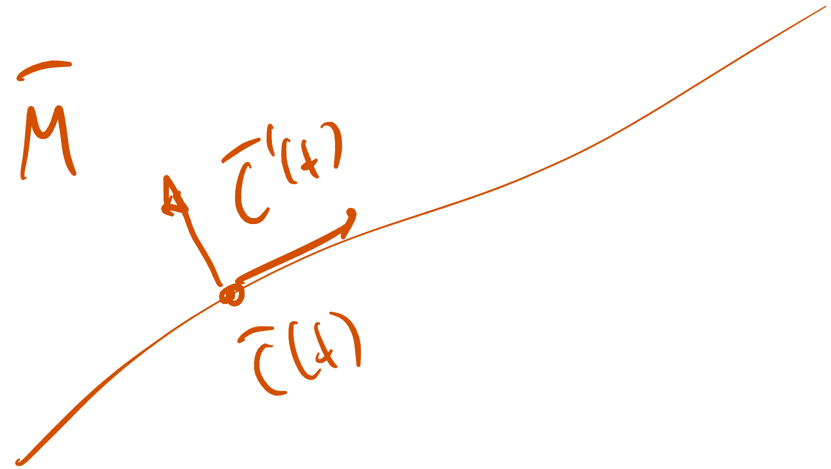
$$c : [0, \ell] \to M \quad , \quad \bar{c} : [0, \ell] \to \bar{M}$$

satisfying

$$\sec_M(P) \leq \sec_{\bar{M}}(\bar{P})$$

<u>whenever</u> $t \in [0, \ell]$, $c'(t) \in P$, $\bar{c}'(t) \in \bar{P}$





Thm 3.19 (Rauch 1951) Under 3.18, and suppose that $\bar{c}(t)$ is **not** conj to $\bar{c}(0)$ (along $\bar{c}|_{[0,t]}$) (for all $t \in [0,\ell]$). Then, if $Y, \bar{Y}$ are J. fields along $c, \bar{c}$ resp. with

$$|Y(0)| = |\bar{Y}(0)| = 0 \quad \text{and} \quad |Y'(0)| = |\bar{Y}'(0)| \neq 0$$

$$Y \perp c', \quad \bar{Y} \perp \bar{c}'$$

$$\implies \quad |Y(t)| \geq |\bar{Y}(t)| \quad \forall t \in [0,\ell]$$

(In particular $c(t)$ not conj. to $c(0)$ (along $c|_{[0,t]}$ for all $t \in [0,\ell]$))

"The trick" consider

$$f(t) = \frac{|Y|^2(t)}{|\bar{Y}|^2(t)}$$

Let us compute, using l'Hôpital's rule

(at $t=0$ equals 0)

$$f(0^+) = \lim_{t \to 0^+} f = \lim_{t \to 0^+} \frac{(|Y|^2)''}{(|\bar{Y}|)''}$$

$-\langle R(Y,c')c', Y\rangle$

J-f eqn

$$\text{Now} \quad (|Y|^2)'' = \langle Y, Y\rangle'' = (2\langle Y', Y\rangle)' = 2\left(\overbrace{\langle Y'', Y\rangle}^{-\langle R(Y,c')c',Y\rangle} + \langle Y', Y'\rangle\right)$$

$$\Rightarrow \quad |Y^2|''(0^+) = 2|Y'|^2(0^+)$$

$$\text{Similarly} \quad |\bar{Y}^2|''(0^+) = 2|\bar{Y}'|^2(0^+)$$

$$\Rightarrow \boxed{f(0^+) = 1}$$

It is enough to show $f'(r) \geq 0 \quad \forall r \in (0,\ell)$

So, let's compute

$$f'(r) = \frac{2\langle \gamma'(r), \gamma(r)\rangle |\bar{\gamma}(r)|^2 - 2\langle \bar{\gamma}'(r), \gamma(r)\rangle |\gamma(r)|^2}{|\bar{\gamma}(r)|^4}$$

$\bar{a} = |\bar{\gamma}(r)|, \quad a = |\gamma(r)|$

$$f'(r) \geq 0 \iff \langle \gamma', \gamma\rangle(r)\, \bar{a}^2 - \langle \bar{\gamma}', \bar{\gamma}\rangle(r)\, a^2 \geq 0$$

$$\overset{(\star)}{\iff} I_r(\gamma,\gamma)\, \bar{a}^2 - \bar{I}_r(\bar{\gamma},\bar{\gamma})\, a^2 \geq 0$$

$$\iff I_r(\underbrace{\bar{a}\gamma}_{Z}, \underbrace{\bar{a}\gamma}_{Z}) - \bar{I}_r(\underbrace{a\bar{\gamma}}_{\bar{Z}}, \underbrace{a\bar{\gamma}}_{\bar{Z}}) \geq 0$$

But now, if I call $Z = \bar{a} Y$, $\bar{Z} = a \bar{Y}$

$$|Z(0)| = |\bar{Z}(0)| = 0$$

$$|Z(r)| = \bar{a}|Y(r)| = \bar{a} a = a|\bar{Y}(r)| = |\bar{Z}(r)|$$

[Remark: if $Z, \bar{Z}$ were J.f. along the same geodesic would follow from 1st index lemma, but they're not.

Choose $E_i(t)$ $1 \le i \le \underset{\text{dim}(M)}{\overset{\shortparallel}{m}}-1$ ON, parallel along c $\perp c'(t)$

$\bar{E}_i(t)$ $1 \le i \le m-1$ ON, parallel along $\bar{c}$, $\perp \bar{c}'(t)$

Write $Z(t) = \displaystyle\sum_{i=1}^{m-1} \underbrace{\langle Z, E_i \rangle}_{Z^i(t)} E_i$ (recall $Z \perp c'$)

and put $\overline{X}(t) = \sum_{i=1}^{m-1} z^i(t)\, \overline{E}_i(t) \in \Gamma(\overline{c}^*TM)$

$\overline{X} \perp \overline{c}'$

Take for simplicity (after rotation)

$$E_1(r) \parallel z(r), \quad \overline{E}_1(r) \parallel \overline{z}(r)$$

Let us compare $\overline{X}$ and $\overline{z}$ (along $\overline{c}$)

$$\overline{X}(0) = \overline{z}(0) = 0, \quad \overline{X}(r) = \overline{z}(r) \left(= z^1(r)\, \overline{E}_1(r)\right)$$

$$I_r(z,z) = \int_0^r |z'|^2 - \underbrace{R_M(z,c',z,c')}$$

$\parallel$ for each t $\quad \geq$ Assumption 3.18

$$\geq \int_0^r |\overline{X}'|^2 - \overbrace{R_{\overline{M}}(\overline{X},\overline{c}',\overline{X},\overline{c}')}$$

1st index Lemma $\geq I_r(\bar{t}, \bar{t})$ $\left(\Longrightarrow ☺ \right)$ $\blacksquare$

Cor. 3.20 $M, \bar{M}, c, \bar{c}$ as in Rauch thm

(under 3.18, $\bar{c}(t)$ not conj. to $\bar{c}(0)$ $\forall t \in [0, \ell]$)

$$p = c(0), \quad v = c'(0), \quad \bar{p} = \bar{c}(0), \quad \bar{v} = \bar{c}'(0)$$

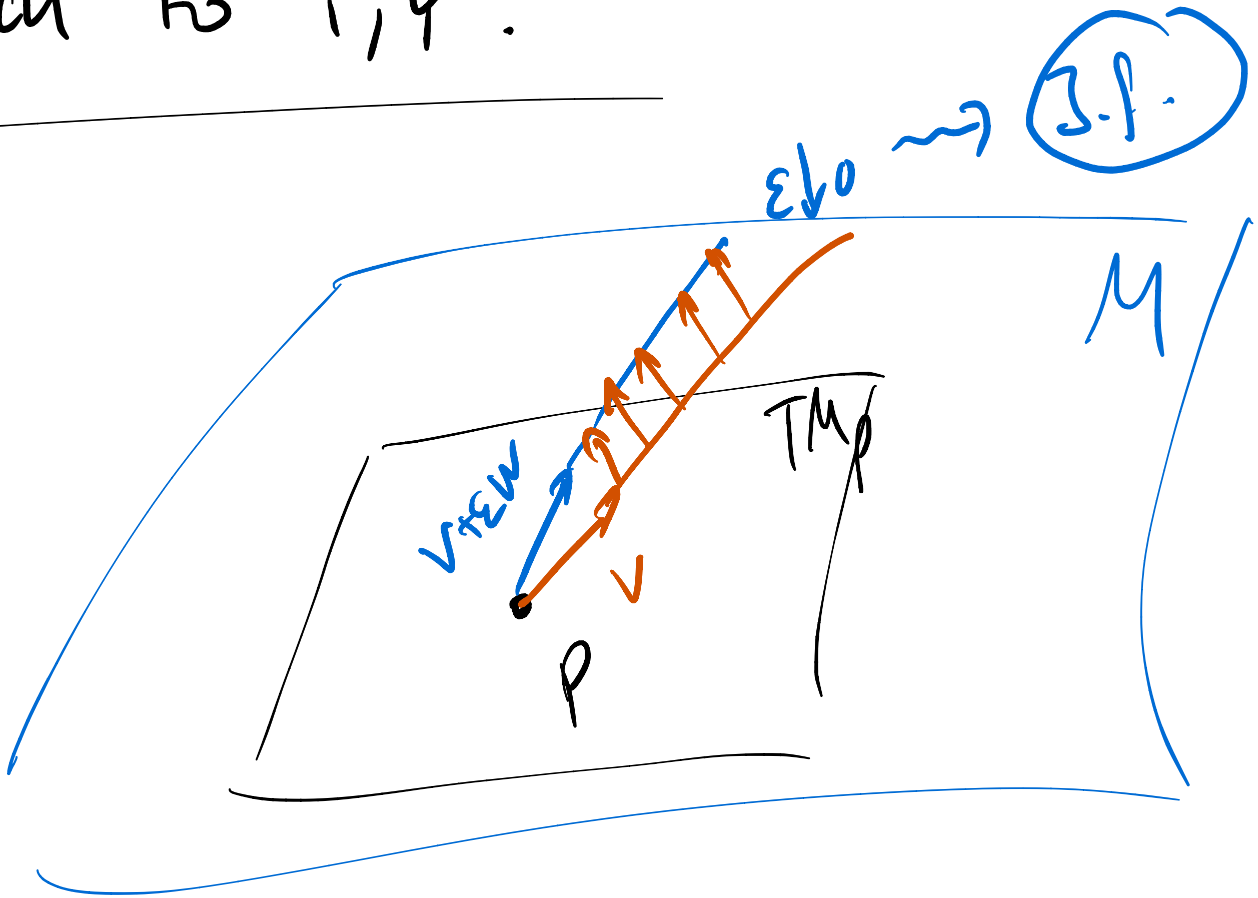
Fix any isometry $H : TM_p \longrightarrow T\bar{M}_{\bar{p}}$ such that $H(v) = \bar{v}$. Then $\forall w \in TM_p$, if $\bar{w} := H(w)$

$$\left| d(\exp_p)_{tv}(w) \right| \geq \left| d(\exp_{\bar{p}})(\bar{w}) \right|$$

proof

$$\left[\begin{array}{ll} Y(t) = d(\exp_p)_{tv}(tw) & \text{is J.f. } (M) \\ \bar{Y}(t) = d(\exp_{\bar{p}})_{t\bar{v}}(t\bar{w}) & \text{is J.f } (\bar{M}) \end{array}\right]$$

Apply Rauch to $Y, \bar{Y}$. $\blacksquare$



Corollary 3.21 M, $\bar{M}$ with same dim

$$\sec_M \leq \kappa \leq \sec_{\bar{M}} \qquad (\text{for some } \kappa \in \mathbb{R})$$

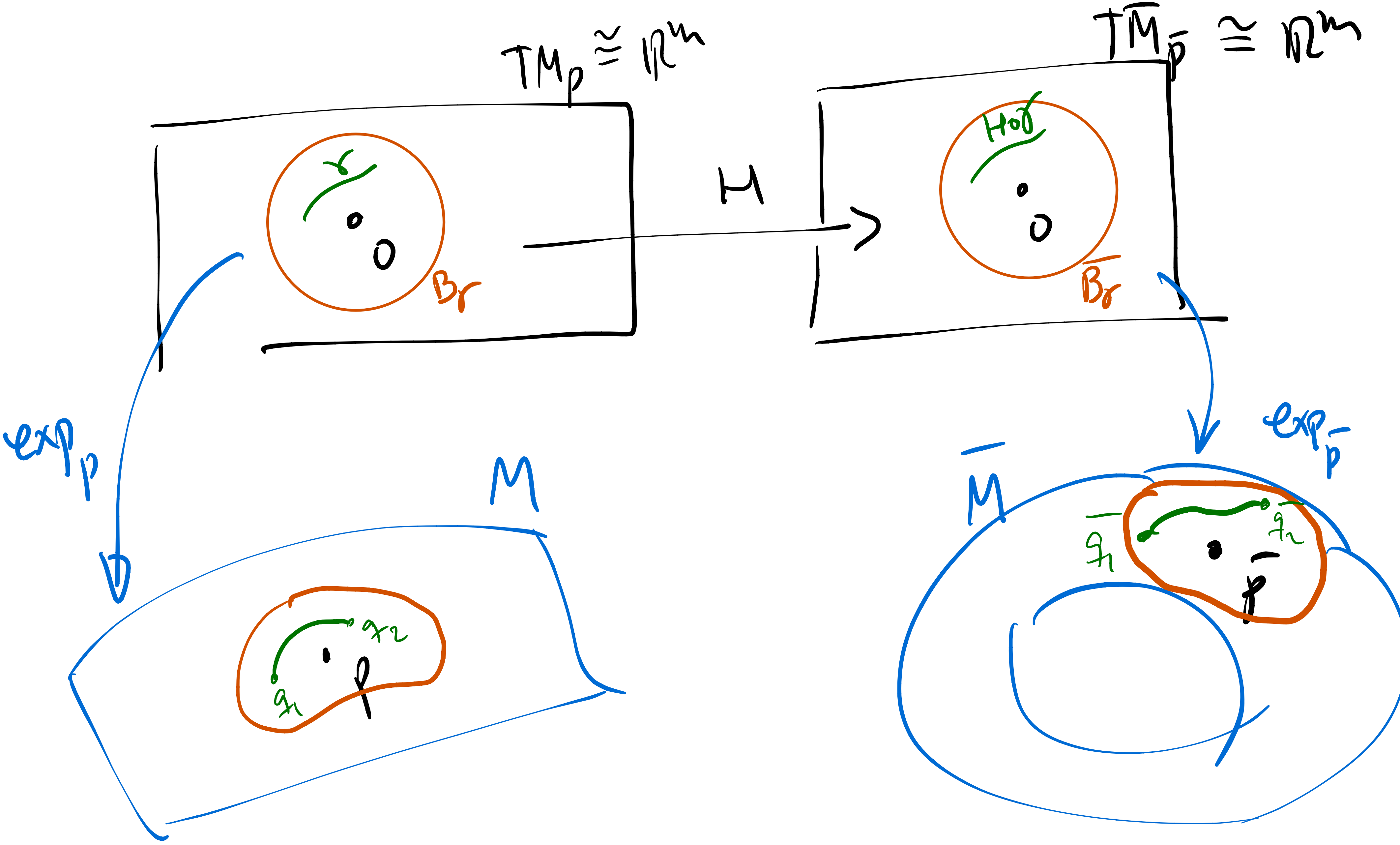
$p \in M$, $\bar{p} \in \bar{M}$ and fix lin. isometry $I : TM_p \to T\bar{M}_{\bar{p}}$

Assume $B_r \subset TM_p$, $\bar{B}_r \subset T\bar{M}_{\bar{p}}$ normal balls

i.e. $\exp_p|_{B_r}$

and $\exp_{\bar{p}}|_{\bar{B}_r}$

are diffeomorphisms



$$L(\exp_p \circ \gamma) \geq \bar{L}(\exp_{\bar{p}} \circ H \circ \gamma)$$

for all $\gamma : [a,b) \longrightarrow B_r$

If $F : B_r(p)^{\subset M} \longrightarrow B_r(\bar{p})^{\subset \bar{M}}$

$$F := \exp_{\bar{p}} \circ H \circ \exp_p^{-1}$$

then F is 1-Lip (i.e $d(F(q_1), F(q_2)) \leq d(q_1, q_2)$)

#

---

proof

chain rule

$$\bar{L}(\gamma \circ H) = \int_a^b \left| d(\exp_{\bar{p}})_{H(\gamma(t))} \circ H(\gamma'(t)) \right| dt$$

$$\leq \int_a^b \left| d(\exp_p)_{\gamma(t)} (\gamma'(t)) \right| dt$$

$$= L(\gamma)$$

$\boxtimes$

Exercise # $\Longrightarrow$ Comparison result for triangles

M $\sec \leq K$ $\bar{M}$ $\sec \geq K$

$$\Longrightarrow \boxed{\bar{c} \leq c}$$

Riemannian coverings (4.3)

$\bar{M}, M$ Rien. mflds of <u>same</u> dim m

smooth map $F : \bar{M} \to M$ is <u>local diffeomorphism</u> if

$\forall p \in \bar{M}$ $\exists$ open nbhd U of p s.t $F|_U$ is diffeo

---

Given $\bar{g}, g$ metrics on $\bar{M}, M$ (resp.), F as above

is <u>local isometry</u> if $F^* g = \bar{g}$

$$\left(\Leftrightarrow \quad \bar{g}_p(v, w) = g_{F(p)}\left(dF_p(v), dF_p(w)\right) \quad \begin{array}{l} \forall p \in \bar{M} \\ \forall v, w \in T\bar{M}_p \end{array} \right)$$

(F is not assumed to be surjective)

Lemma 4.10 $F, G : \bar{M} \longrightarrow M$ local isometries

$\bar{M}$ is connected, If $F(p) = G(p)$ and $dF_p = dG_p$ for one point $p \in \bar{M}$

$\Longrightarrow \quad F \equiv G$

~~proof~~ $\boxed{\text{Key observation}}$ if $F : \bar{M} \longrightarrow M$ is local isom. and

$$\begin{cases} \bar{c} \text{ is unit speed geod. on } \bar{M} \\ c \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{"} \quad \text{on } M \end{cases}$$

such that $\bar{c}(0) = p$, $\bar{c}'(0) = v \in T\bar{M}_p$, $c(0) = F(p)$, $c'(0) = dF_p(v)$

Then $\boxed{c = F \circ \bar{c}}$ (exercise check details by yourselves)

As a consequence,

$$dF_q : T\bar{M}_q \longrightarrow TM_{F(q)} \quad \text{(is linear isometry)}$$

$$\boxed{F \circ \exp_q = \exp_{F(q)} \circ dF_q}$$

$$\begin{tikzcd} T\bar{M}_q \arrow[r, "dF_q"] \arrow[d, "\exp_q"] & TM_{F(q)} \arrow[d] \\ \bar{M} \arrow[r, "F"] & M \end{tikzcd}$$

Put $A = \{ q \in \bar{M} : F(q) = G(q) \text{ and } dF_q = dG_q \}$

- $A \neq \phi$ $(\Leftarrow p \in A)$
- A closed
- A open because if $q \in A$

(b) $F \circ \exp_q = \exp_{F(q)} \circ dF_q \equiv \exp_{G(q)} \circ dG_q = G \circ \exp_q$

$\Longrightarrow$ $F \equiv G$ in nbhd of q

$(\Longrightarrow$ all pts in nbhd belong to $A)$

We conclude (using $\bar{M}$ connected) $A = M \overset{(b)}{\Longrightarrow} F \equiv G$ ∎

$\boxed{\text{Covering map}}$ (topological spaces) M, $\overline{M}$ top. spaces

$$\pi : \overline{M} \longrightarrow M \quad \underline{\text{covering map}}$$

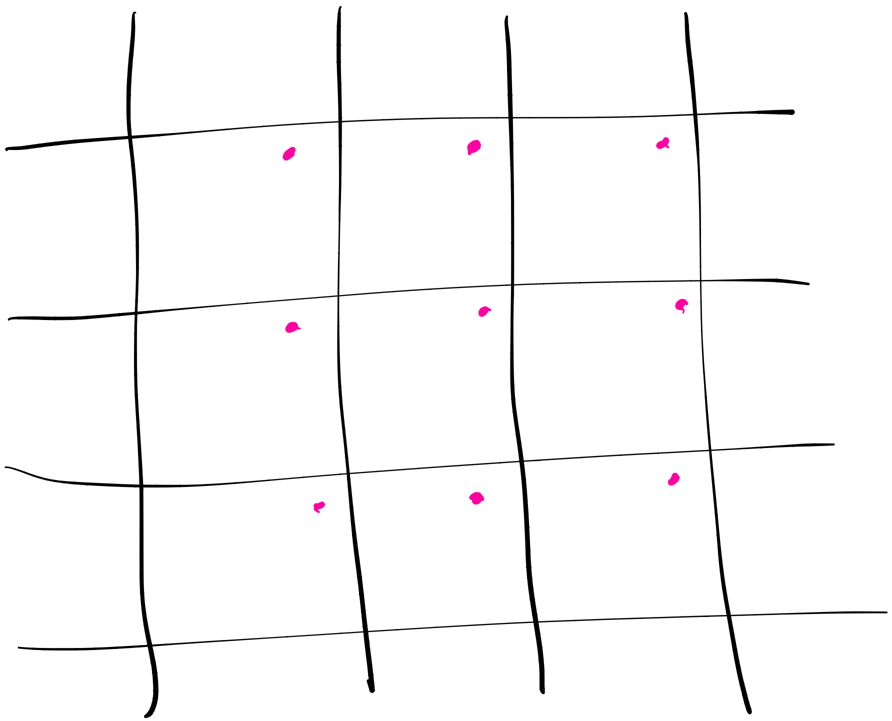
if <u>continuous</u>, <u>surjective</u>, $\forall q \in M$, $\exists U_q$ open nbd s.t $\pi^{-1}(U_q)$ union of <u>pairwise disjoint</u> <u>open sets</u> s.t the restiction of π on <u>each of them</u> is homeom.

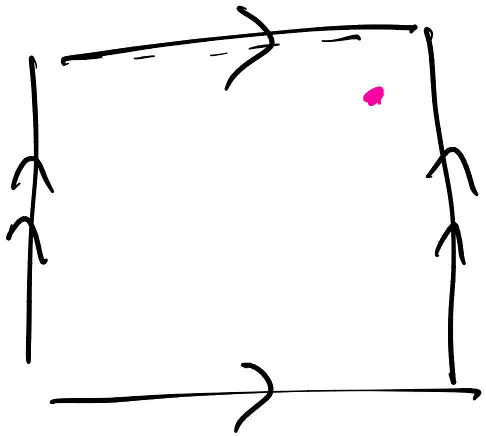
<u>Defin 4.11</u> $(\overline{M}, \overline{g})$, (M, g) Riem. mflds same dim m

A smooth covering map $\pi : \overline{M} \to M$ s.t $\pi^* g = \overline{g}$ is called <u>Riem. covering map</u>

<u>Example</u>

$$\mathbb{R}^2 \xrightarrow{\quad F \quad} \mathbb{R}^2/\mathbb{Z}^2$$





Prop 4.12 $\overline{M}$ complete Riem. mfld, M connected

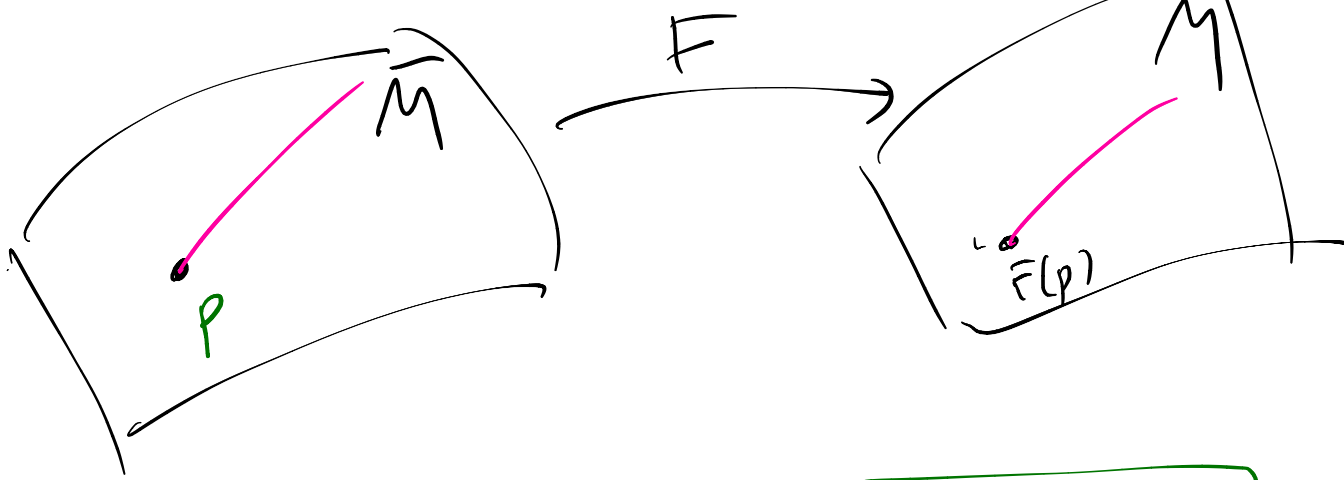
Then every local isom. $F : \overline{M} \longrightarrow M$ is Riem. covering map.

proof **Step 1**. F is surjective

$\overline{M}$ complete
M connected
F local isom.

$\Downarrow$

M complete
F surjective

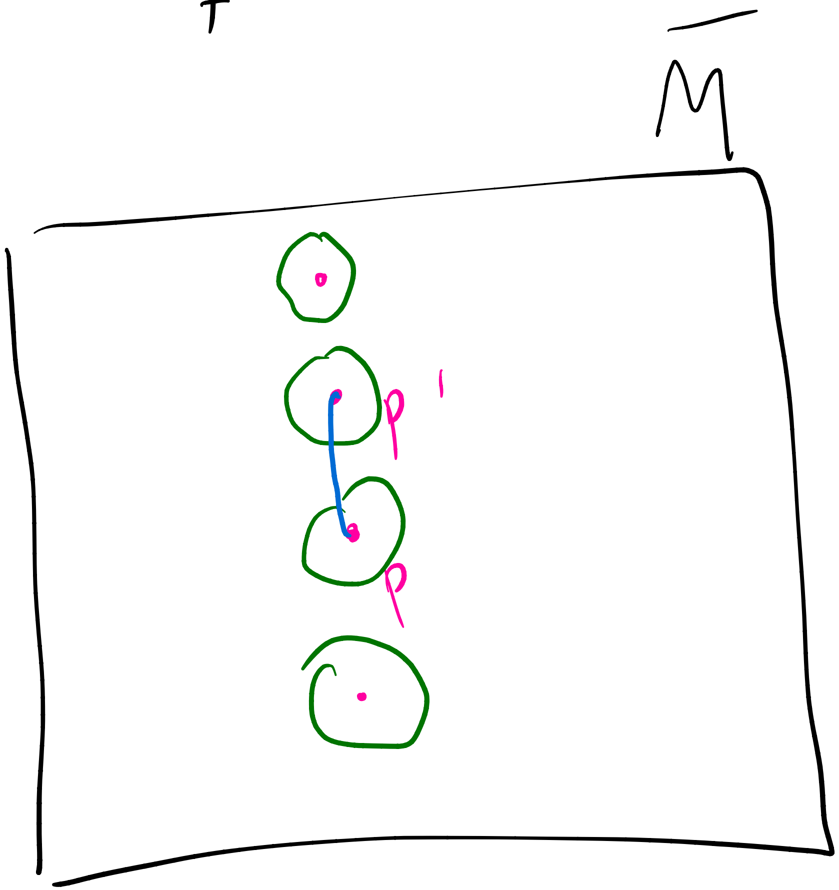


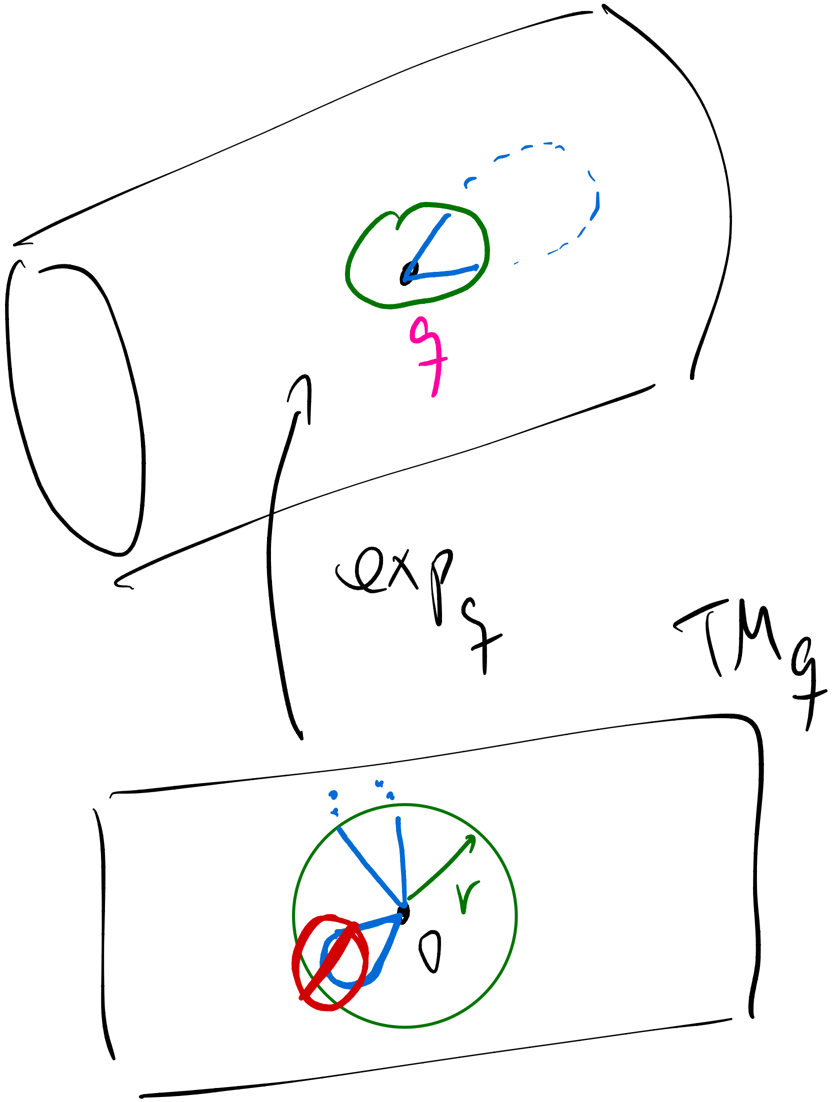
$F(\overline{M}) \subset M$ complete $\Rightarrow$ $F(\overline{M})$ closed

F local isometry $\Rightarrow$ $F(\overline{M})$ open

M connected $\Rightarrow$ $F(\overline{M}) = M$

Step 2 Let us show $\forall q \in M \ \exists U_q$ st $F^{-1}(U_q)$ disjoint union of open set mapped diff. (and isom.) onto U_q.





Chose $r>0$: $\exp_q \big|_{B_r(0)}$ is diffeo.

$\overline{M}$ complete $\Rightarrow$ $\exists$ minimizing geodesic $\boxed{\bar{c}}$ joining p, p'

Let $c = F \circ \bar{c}$ is a geodesic loop joing p with p again

since $B_q(r)$ normal ball, c must exit $B_r(q)$ and come back

exercise Local isometry 1-Lip
$d(F(p), F(q)) \leq d(p,q)$

$$\Rightarrow L(c) \geq 2r$$

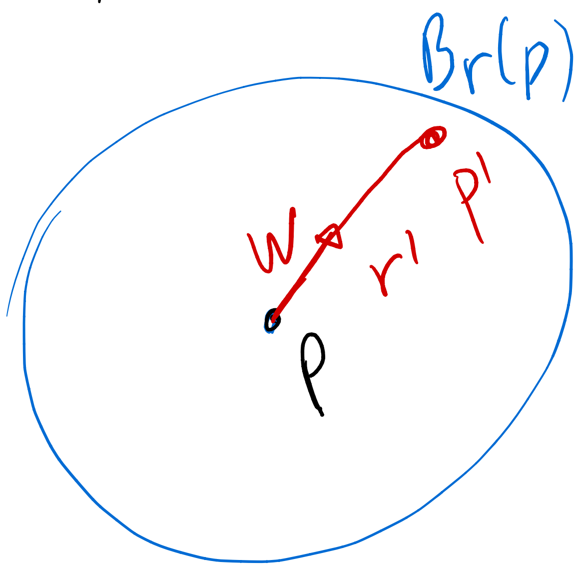
$$\Rightarrow L(\bar{c}) \geq L(F \circ \bar{c}) = L(c) \geq 2r$$

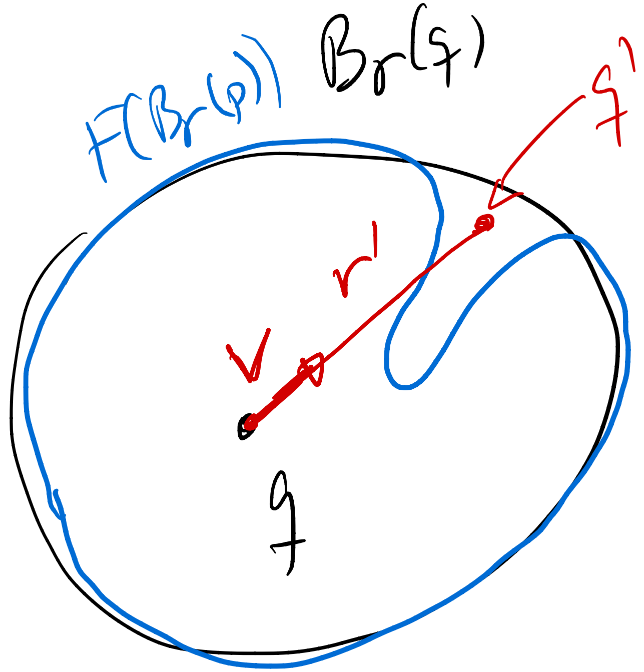
$$\Rightarrow B_r(p') \cap B_r(p) = \emptyset$$

Let us show $\forall p \in F^{-1}\{q\}$ $F|_{B_r(p)}$ maps $B_r(p)$ diffeom onto $B_r(q)$

F isometry $\Rightarrow$ $F(B_r(p)) \subset B_r(q)$
($\Rightarrow$ 1-Lip)

$$\left(dF_q(w) = v \right)$$





Let us prove $\supset$

Similar argument as in Hopf Rinow

$\blacksquare$

Review of covering maps

$F : \bar{M} \longrightarrow M$ covering map

$\gamma \in \mathrm{Homeo}(\bar{M})$ is <u>deck transformation</u> when $F \circ \gamma = F$

The group Γ of all deck transf. acts <u>freely</u> and <u>prop. discontinuously</u> on $\bar{M}$

no fixed pts

if $K \subset \bar{M}$ is compact set $\{\gamma : \gamma(K) \cap K \neq \phi\}$ is finite

$\pi_1(\bar{M})$ is <u>subgroup</u> $\pi_1(M)$. When it is a normal subgroup then $\Gamma \cong \pi_1(M) / \pi_1(\bar{M})$ and $M \cong \bar{M} / \Gamma$ (*)

(*) $\tilde{M}$ $\quad p \equiv q \iff \exists \gamma \in \Gamma$ s.t $\gamma(p) = q$

If F is Riem. covering map then deck transf are isometries (of $\tilde{M}$) (exercise)

$\boxed{F \circ \gamma = F}$

- γ is bijective (it is homeo.)
- γ is isometry $\langle d\gamma_p(v), d\gamma_p(w) \rangle = \langle w, v \rangle$

show this using $\boxed{}$ and chain rule

(recall that F is a local isometry)

Models of "the" m-dim hyperbolic space $(\mathbb{H}^m)$ $\quad K = -1$

(1) $x^1, \dots, x^m$ coordinates $\mathbb{R}^m$

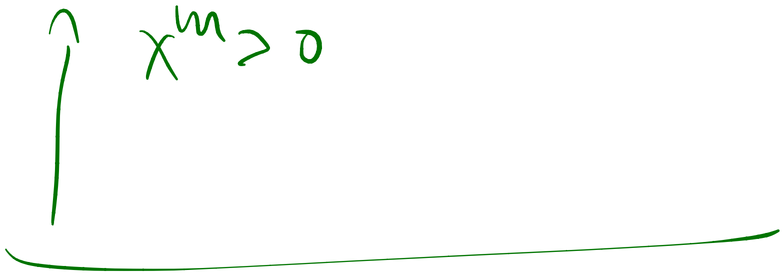
$M = \{x^m > 0\}$ $\qquad \tilde{g}_{ij} = \dfrac{\delta_{ij}}{(x^m)^2} \lambda^2$

$$L(c) = \int_a^b \sqrt{\tilde{g}_{ij} \dot{c}^i \dot{c}^j}$$
$$= \lambda \int_a^b \sqrt{g_{ij} \dot{c}^i \dot{c}^j}$$

$v, w \in TM_x \cong \mathbb{R}^m$

$(\lambda^2 = |K|^{-1})$

$$\langle v, w \rangle = \frac{v^i v^j \delta_{ij}}{(x^m)^2} \qquad (\#)$$



<u>exercise</u> $(\#) \rightsquigarrow$ Christoffels $\rightsquigarrow R_{ijk\ell}$

show that sec $= -1$

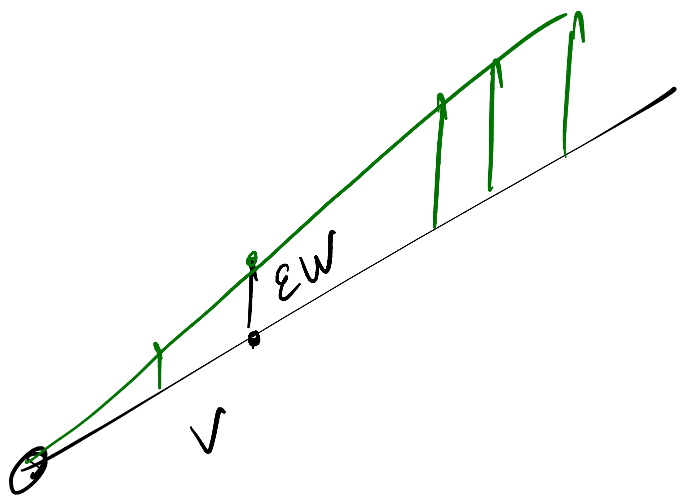
similar model Poincaré disk (see exercises)

② Suppose M is complete, $\sec = -1$, simply connected

Fix $p \in M$, consider $\exp_p : \mathbb{R}^n \cong TM_p \to M$

Fix isometry H from $(\mathbb{R}^n, g_{Eucl})$ to (TM_p, g_p)

$t \mapsto d(\exp_p)_{tv}(tw)$ is J.f.



$\Rightarrow$ it satisfies J.f. eq'n

$$Y'' - Y = 0$$

$(\text{if } Y \perp c')$

$w \perp v$

$$\rightsquigarrow \quad d(\exp_p)_{tv}(tw) = \sinh(t)\, w$$

$\rightsquigarrow$ "$g = g_{ij}$" "pullback" of g by $\exp_p \circ H$

$v, w \in \mathbb{R}^m$

$$g(w,w) = \left(w \cdot \frac{x}{|x|}\right)^2 + \left(|w|^2 - \left(w \cdot \frac{x}{|x|}\right)^2\right) \frac{\sinh^2 |x|}{|x|^2}$$

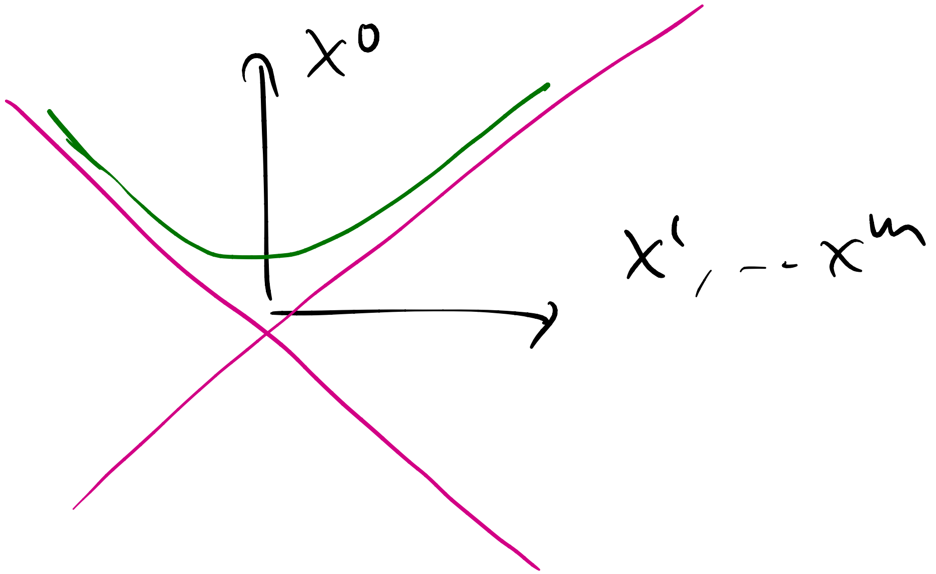
($\rightsquigarrow$ $g(v,w)$ by parallelogram id)

---

(3) $(x^0, x^1, \dots x^m)$ $(m=3)$

$$\langle v, w \rangle = v^0 w^0 - v^1 w^1 - v^2 w^2 - v^3 w^3 \quad (m=3)$$

$$\|v\|^2 = \langle v, v \rangle$$



$\{x \in \mathbb{R}^{m+1} : \langle x, x \rangle = 1 \}$
$= \lambda^2$

equipped $\langle \; \rangle$ is a Riem. manifold.

Rem we can prescribe curvature $K < 0$ insted -1

Define $\mathbb{M}^m_K = \begin{cases} \text{Sphere radius } K^{-1/2} & \text{if } K > 0 \\ \mathbb{R}^m \text{ Euclidean} & \text{if } K = 0 \\ \text{Hyperbolic space} & \text{if } K < 0 \end{cases}$

(choose your favorite model as above!)

<u>Thm 4.13</u> (Killing 1891, Hopf 1926) M is m-dim space form (connected, compl. Riem. mfld with sec. curv. $\equiv \kappa \in \mathbb{R}$).

Then $\exists$ group $\Gamma \subset \mathrm{Isom}(\mathbb{M}^m_\kappa)$ acting freely and prop. disc. on $\mathbb{M}^m_\kappa$ s.t

$$M \cong \mathbb{M}^m_\kappa / \Gamma$$

Moreover if M is simply connected then $M \cong \mathbb{M}^m_\kappa$

<u>proof</u> $\kappa < 0$ $\boxed{\kappa = -1}$

Fix $p \in \mathbb{H}^m (= \mathbb{M}^m_\kappa)$ and $q \in M$, and

linear isometry $H : T\mathbb{H}^m_p \to TM_q$

Define
$$F := \exp_q \circ H \circ (\exp_p)^{-1} : \mathbb{H}^m \longrightarrow M$$

- $\exp_p : T\mathbb{H}^m_p \longrightarrow \mathbb{H}^m$ is a diffeomorphism

  $\sec \leq 0 \xRightarrow{\text{Cor 3.20}}$ no conjugate pts (i.e no critical pts for $\exp_p$)

- $\exp_q$ is an immersion (by same reason)

- F is local isometry (by cor. 3.21)

- F covering map (by Prop 4.12)

$\Rightarrow$ Group Γ of deck transf of F acts on $\mathbb{H}^m$ freely and prop. discontinuously

Since $\mathbb{H}^m$ is simply connected

$$\{e\} = \pi_1(\mathbb{H}^m) \overset{\triangleleft}{\subset} \pi_1(M)$$

$$F : \mathbb{H}^m \longrightarrow M$$

$$\Rightarrow \quad M \cong \mathbb{H}^m / \Gamma \quad \text{and} \quad \Gamma \cong \pi_1(M) = \pi_1(M) / \pi(\mathbb{H}^m)$$

In part if M is simply connected $\pi_1(M) = \{e\}$

$$\Gamma = \{id\} \Rightarrow M \cong \mathbb{H}^m$$

$\boxed{k=0}$ the same changing $(\mathbb{H}^m, g^{hyp})$ by $(\mathbb{R}^m, g^{Eucl})$

$\boxed{k=1}$ Only difference: we need to deal with conjugate points

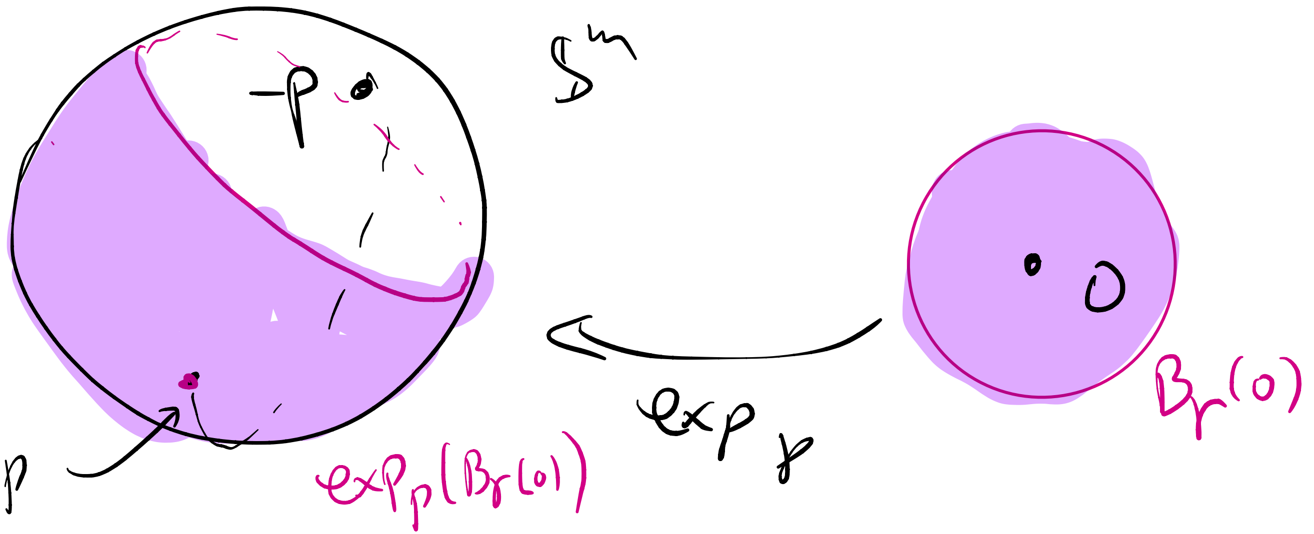
p is antipodal to $-p$

Given $p \in S^m \subset \mathbb{R}^{m+1}$ and $q \in M$, and linear isom.

$H : TS^m_p \longrightarrow TM_q$

define $F := \exp_q \circ H \circ \Big(\underbrace{\exp_p\big|_{B_\pi(0)}}_{\text{diffeomorphism!}}\Big)^{-1} : (S^m \setminus \{-p\}) \longrightarrow M$

radius $= 1$



As for $K=-1$, F is a local isometry from $S^m \setminus \{-p\} \longrightarrow M$

Choose $\tilde{p} \in S^m \setminus \{p, -p\}$ and define

$$\tilde{F} := \exp_{\tilde{q}} \circ \tilde{H} \circ \left(\exp_{\tilde{p}}\big|_{B_\pi(0)}\right)^{-1} : (S^m \setminus \{-\tilde{p}\}) \to M$$

with $\tilde{q} = F(\tilde{p})$, $\tilde{H} = dF_{\tilde{p}}$

$$\tilde{F}(\tilde{p}) = \tilde{q} = F(\tilde{p}), \quad d\tilde{F}_{\tilde{p}} = \tilde{H} = dF_{\tilde{p}}$$

(both $F, \tilde{F}$ are local isometries $S^m \setminus \{-\tilde{p}, -p\} \longrightarrow M$

$$\xRightarrow{\text{Lemma 4.10}} \quad F \equiv \tilde{F}.$$

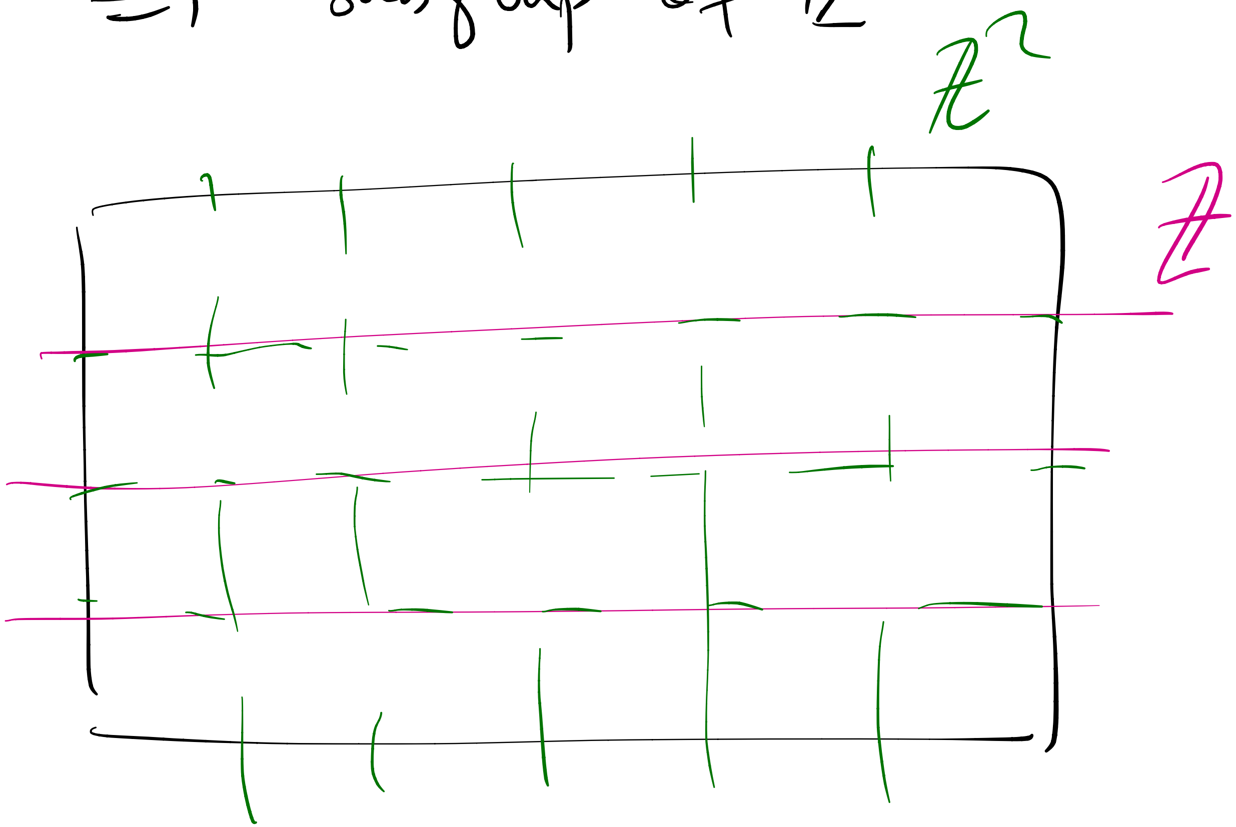
The union of the domains of F and $\tilde{F}$ cover S^m $\Rightarrow$

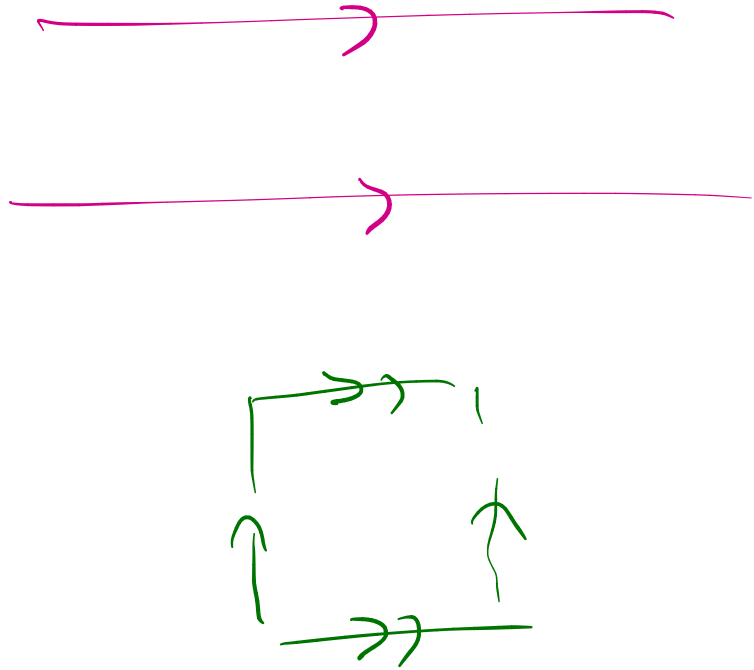
$F, \tilde{F}$ define a local isom $S^n \to M$

and we conclude as in $K=-1$ or $K=0$ ∎

<u>Examples</u> 1. $K=0$ $\Gamma \subset \mathrm{Isom}(\mathbb{R}^n, g^{\mathrm{Eucl}})$

$\Rightarrow$ "subgroup of $\mathbb{Z}^k$"





2. <u>Hyperbolic space forms</u>: Every compact oriented surface of genus ≥ 2 can be realized as a quotient of

$$(\mathbb{H}^2, g^{hyp}) \big/ \Gamma \qquad \text{(ref's in the notes)}$$

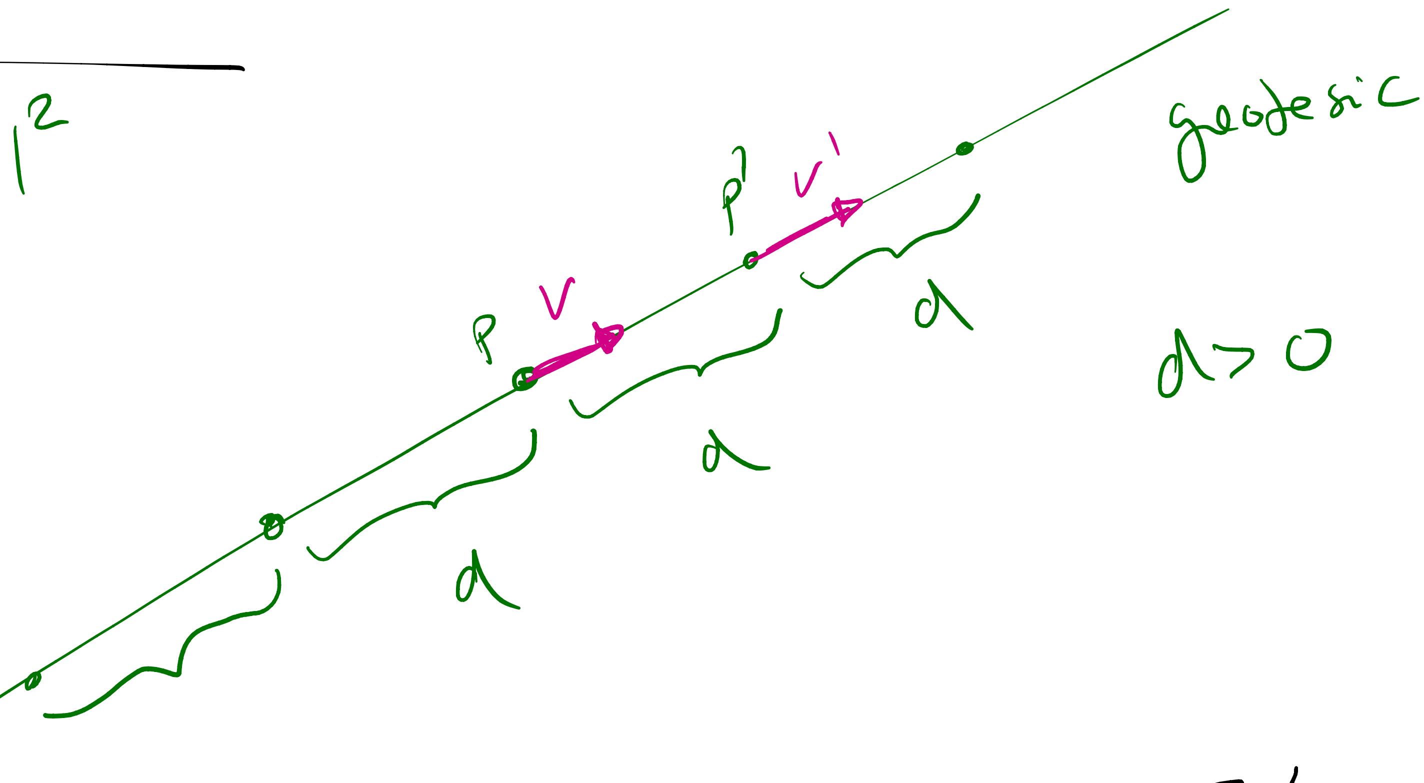
---

hyperbolic $\quad \mathbb{H}^2$

$\exists\, \gamma$ isometry $\mathbb{H}^2$

sending $p \to p'$

$(d\gamma_p(v) = v')$ $\quad v \to v'$



$\exp_{p'} \circ M \circ (\exp_p)^{-1}$

$M : T\mathbb{H}^2_p \to T\mathbb{H}_{p'}$

$$\Gamma = \{ \gamma^k ,\ k \in \mathbb{Z} \}$$

you can consider $\mathbb{H}^2/\Gamma$ is a space form with $K=-1$ and fund. group $\mathbb{Z}$.

---

3. spherical space forms S^m/Γ

m odd $\quad S^m = S^{2n-1} \subset \mathbb{R}^{2n} = \mathbb{C}^n$

$$(z_1, \dots, z_n) \longmapsto \left(e^{2\pi i k q_1/p} z_1, \dots, e^{2\pi i k q_n/p} z_n\right)$$

q_i, p coprime integers.

Compare with $\quad S^2 \subset \mathbb{C} \times \mathbb{R} \quad (z, x^3) \longmapsto (e^{2\pi/p\, i} z, x^3)$

Remark: the only isometries without fixed pts from $S^m \to S^m$ if m is even are

$$p \longmapsto -p$$

Immediate Corollary:

Thm 4.17 If M is space form with $K>0$ and even dim, then M is isometric to S^m or $\mathbb{R}P^m = S^m/\sim \quad (p \sim -p)$

(If M is orientable $\Rightarrow M = S^m$)

Hadamard mflds

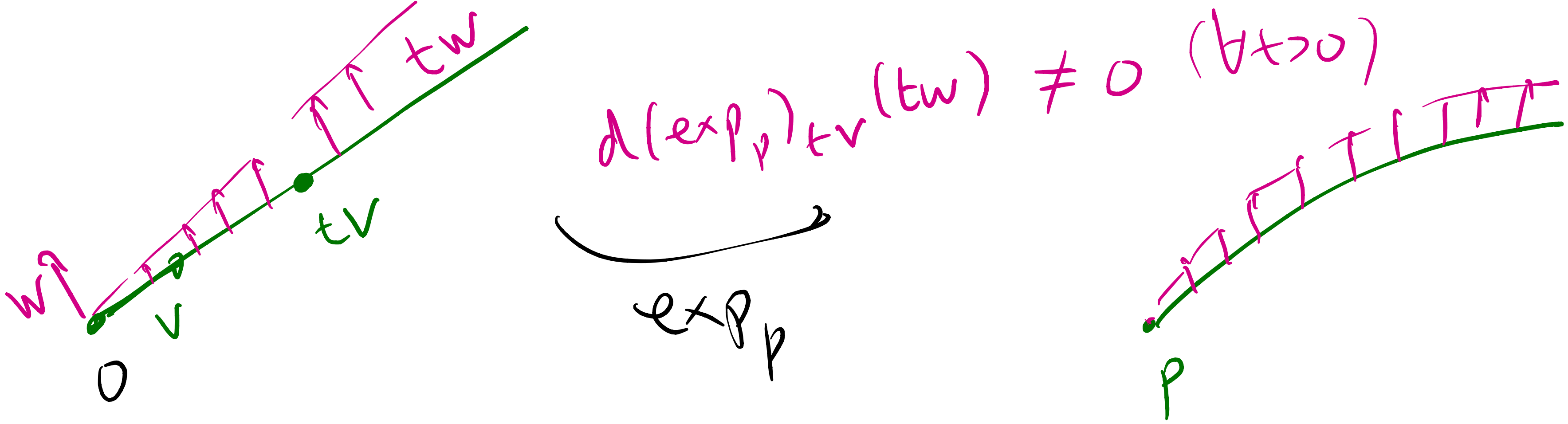
Thm 4.18 (Hadamard–Cartan) (M, g) m-dim complete Riem. mfld, sec curv ≤ 0. Let $p \in M$.

Then $\exp_p : TM_p \to M$ is a covering map.

In particular if M is simply connected then $\exp_p$ is a diffeo (and M is diffeo to $\mathbb{R}^m$)

---

proof: sec ≤ 0, by comparing M and $(\mathbb{R}^m, g^{Eucl})$ (using Rauch) there are no conjugate points along any given geodesic



$\Rightarrow \quad d(\exp_p)_v(w) \neq 0 \qquad \forall v \in TM_p \quad \forall w \neq 0$

$\Leftarrow \quad \exp_p$ is local diffeo

Endow TM_p with metric

$\bar{g} := \exp_p^* g$, i.e. $\bar{g}(w_1, w_2) := g(d\exp_p(w_1), d\exp_p(w_2))$

$(TM_p, \bar{g})$ is complete (Hopf-Rinow)

$\Longrightarrow$ $\exp_p$ is a covering map.

$\bar{M} := (TM_p, \bar{g})$ is simply connected $\quad M = \bar{M}/\Gamma$

If M is simply connected $\Rightarrow \Gamma = \{id\}$

$\Rightarrow \bar{M} = M$

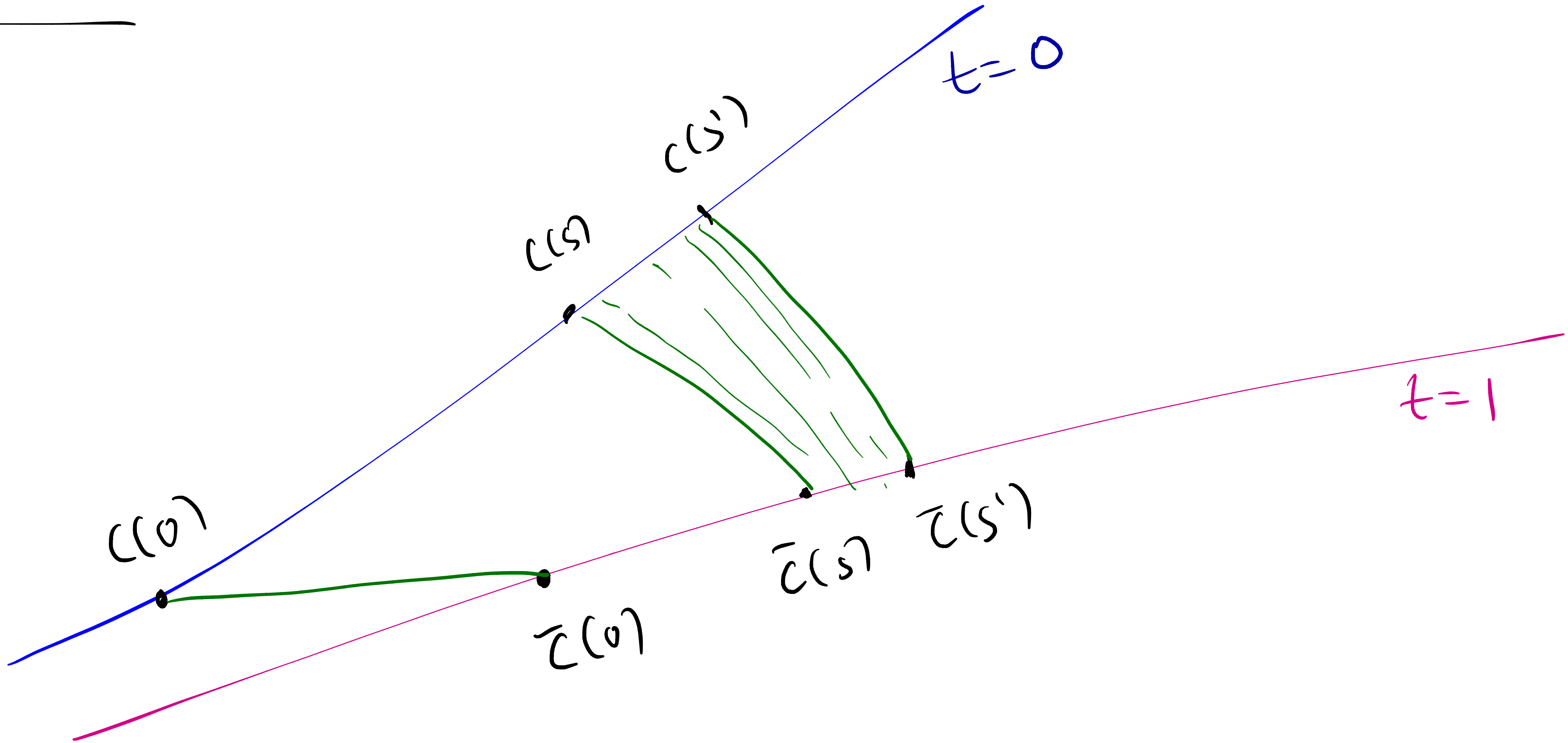
$\boxtimes$

<u>Def'n</u> <u>Hadamard mfld</u> is a complete, simply connected, Riem. mfld with ≤ 0 sec curv.

<u>model</u> $\quad (\mathbb{R}^n, \bar{g})$

Lemma 4.19 M Hadamard manifold, then for any given pair of geodesics $c, \bar{c}$ (constant speed maybe different)

$$\mathbb{R} \ni s \longmapsto h(s) := d(c(s), \bar{c}(s)) \text{ is convex}$$



Let $\gamma_s : [0,1] \longrightarrow M$ be <u>the</u> geodesic joining $c(s)$ and $\tilde{c}(s)$.

$$h(s) := L(\gamma_s) \qquad \left[\text{ notice } \gamma_s = \gamma_s(t) = \gamma(s,t) \right]$$

Using 2nd variation formula (Thm 3.1) $\left(V_{s_0} = \gamma_* \frac{\partial}{\partial s}\Big|_{s=s_0} \right)$

$$\left({}' = \frac{d}{dt} \right)$$

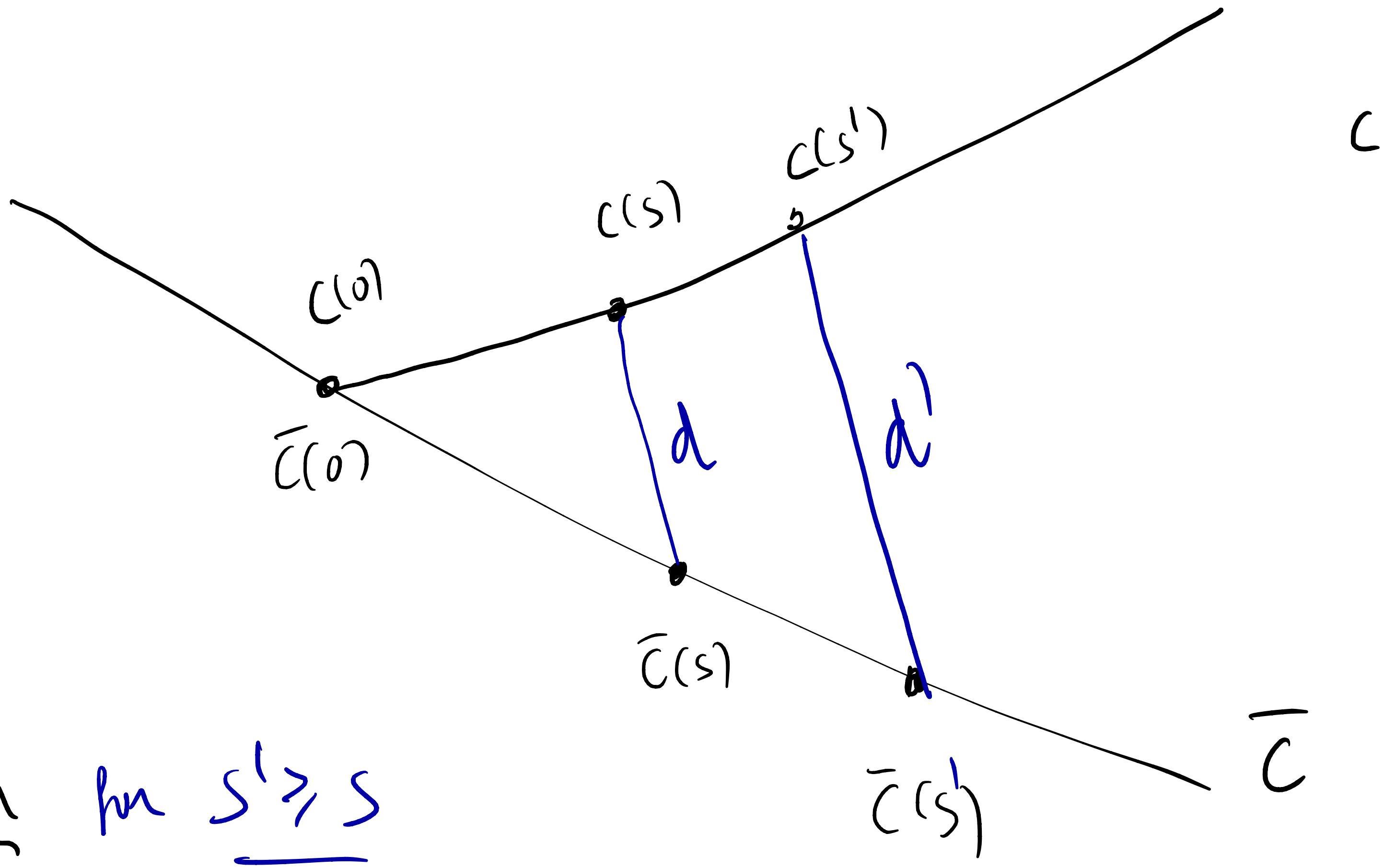
$$h''(s_0) = \frac{d^2}{ds^2}\Big|_{s=0} L(\gamma_{s_0+s}) = \int_0^1 \overset{\substack{0 \\ \wedge\!|}}{|(V_{s_0}')^\perp|^2} \underset{=}{-} \overbrace{R(V_{s_0}, \gamma'_{s_0}, V_{s_0}, \gamma'_{s_0})}^{\substack{0 \\ \wedge\!|}}\, dt$$

$$+ \underbrace{\text{" end point terms "}}_{= 0}$$

$$\boxed{\frac{D}{\partial s} V_s(0) = \frac{D}{ds} c' = 0} \qquad \left(\text{sim. } \frac{D}{\partial s} V_s(1) = 0 \right)$$

$\blacksquare$

Rem Lem 4.19 $\Rightarrow$ "Thales thm"

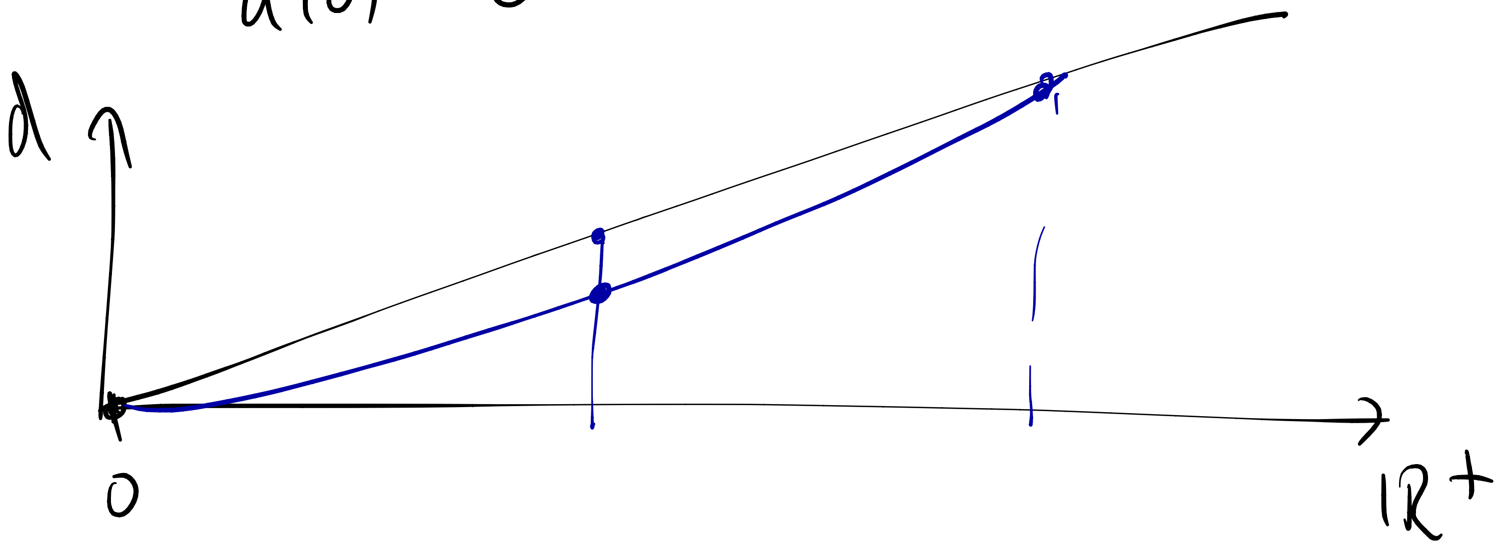


Then for $s' \geq s$

$$\frac{d}{d'} \leq \frac{s}{s'} = \frac{d(c(s), c(0))}{d(c(s'), c(0))} = \frac{d(\bar{c}(s), c(0))}{d(c(s'), c(0))}$$

proof $\quad d(s) := d(c(s), \bar{c}(s))$ convex $\mathbb{R}^+ \to \mathbb{R}^+$

$d(0) = 0$

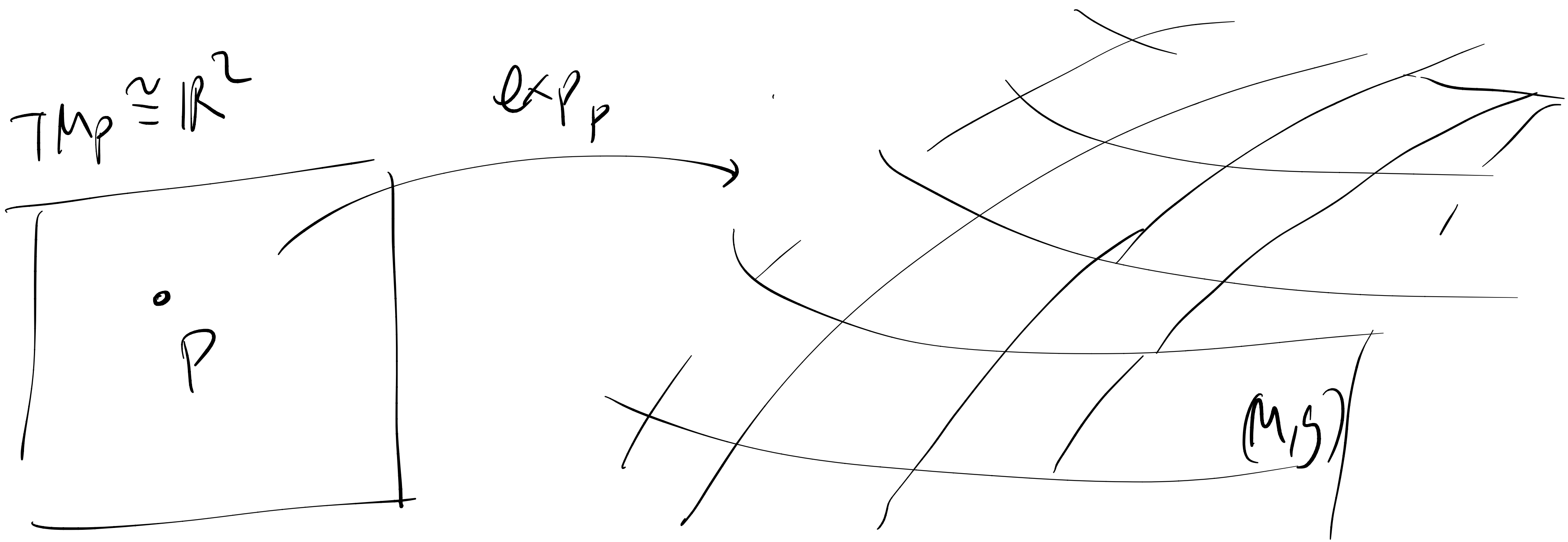


∎

There are "plenty" of Hadamard mflds

2D Example

Given a Hadamard manifold of dim 2, M, consider normal coordinates at TM_p, for some given $p \in M$



$$\bar{g} = (\exp_p)_* g$$

Use polar coordinates in TM_p, fix $H : (\mathbb{R}^2, g^{Eucl}) \longrightarrow (TM_p, g_p)$ isometry

$\theta \in \mathbb{R}$ (or $[0, 2\pi)$)

$r > 0$

normal coord $\begin{cases} x^1 = r\cos\theta \\ x^2 = r\sin\theta \end{cases}$

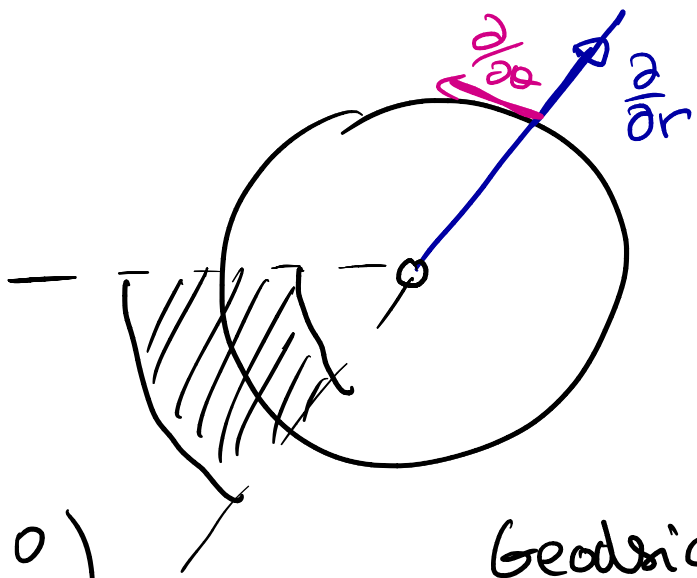
(r, θ) are by def'n polar (normal) coord.

---

by Gauss Lemma

- $\bar{g}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 1$
- $\bar{g}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right) = 0$

$TM_p \setminus \{0\}$



$\Longrightarrow$ $\begin{matrix} a < r < b \\ c < \theta < d \end{matrix}$ $\bar{g}_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}$ $\longrightarrow$ Geodesic parallel coordinates

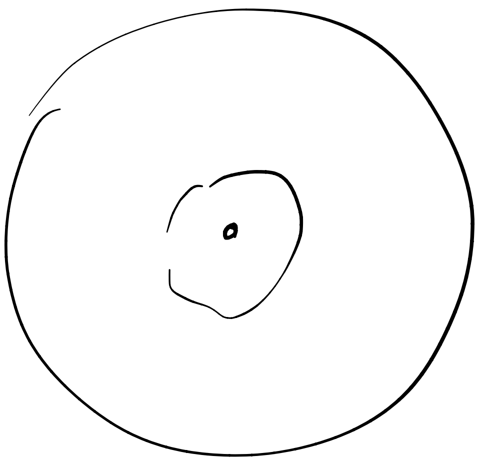
$E = E(r, \theta)$

$$\boxed{K = \frac{-(\sqrt{E})_{rr}}{\sqrt{E}}}$$

Therefore $K \leq 0 \iff (\sqrt{E})_{rr} \geq 0$ (*)

This allows us to produce models of 2D Hadamard manifolds

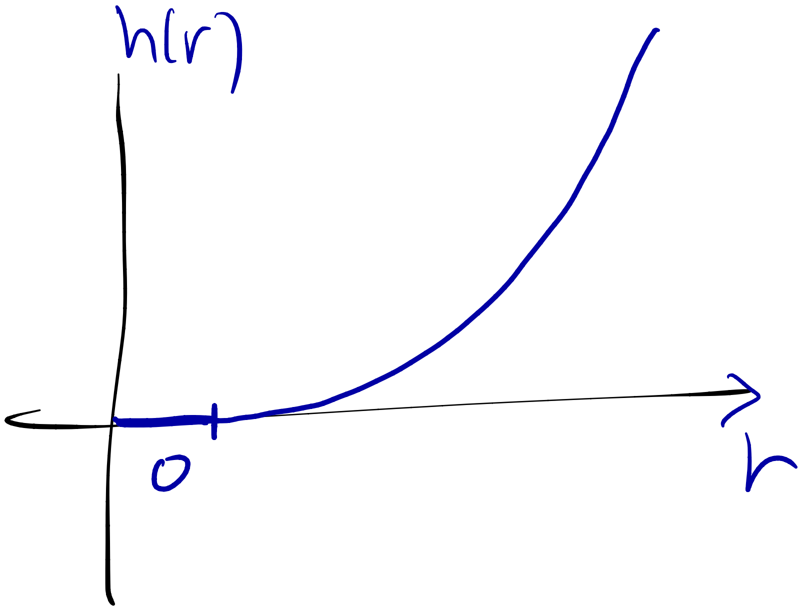
$M := (\mathbb{R}^2, \bar{g})$, with $\bar{g}$ s.t in polar coordinates satisfies



$$\longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & r^2 + h(r,\theta) \end{pmatrix}$$

Euclidean metric in polar coord

$$h(r) = \begin{cases} 0 & r < \varepsilon \\ \text{convex in } r, & r \geq \varepsilon \end{cases}$$



Prop 4.20 M Hadamard, $c, \bar{c}$ are two geodesic with $c(\mathbb{R}) \neq \bar{c}(\mathbb{R})$ and $\sup_{s \in \mathbb{R}} d(c(s), \bar{c}(s)) < \infty$

The two geodesics bound a flat strip (isometric to $[0,a] \times \mathbb{R}, g^{Eucl}$) is a totally geodesic submanifold

$M \subset \overline{M}$ is a totally geodesic submanifold $\iff$

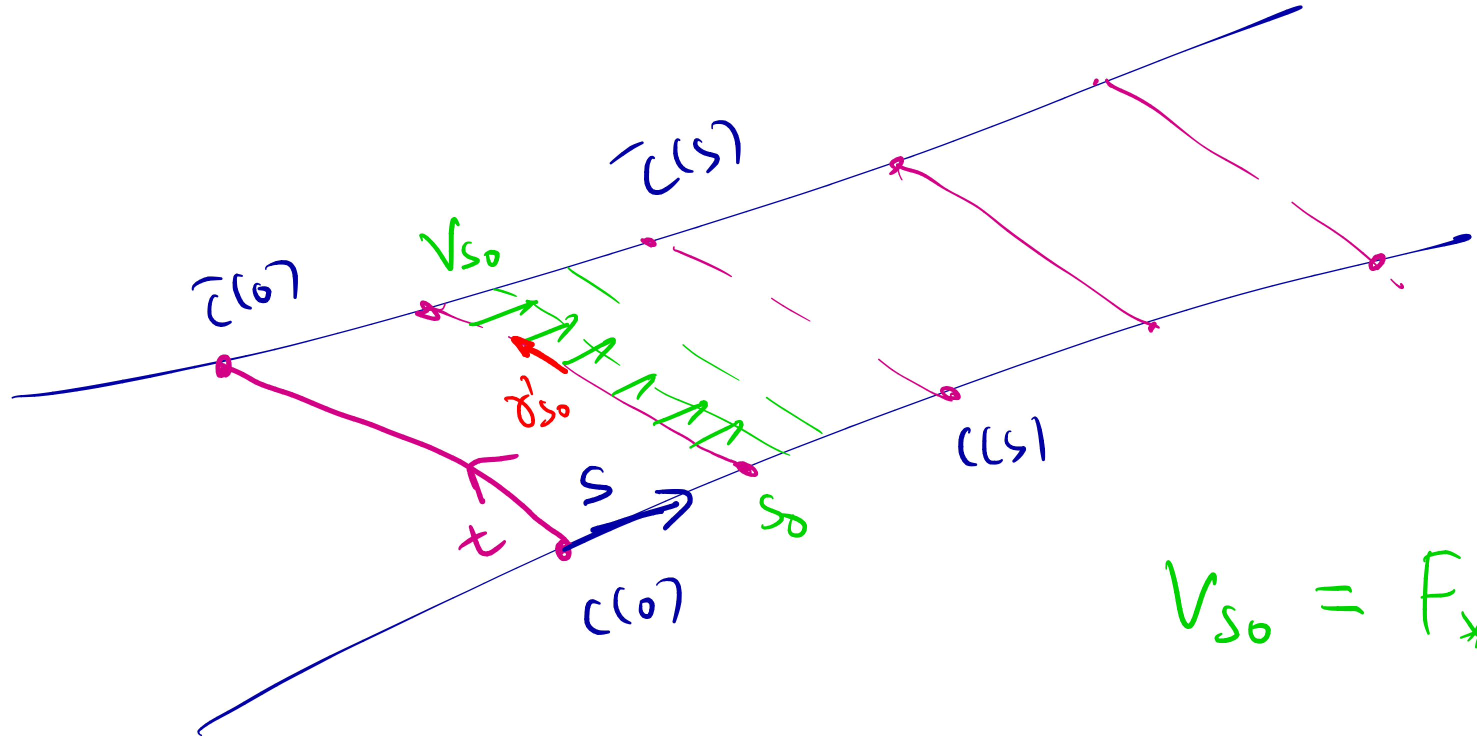
- 2^{nd} FF $\equiv 0$
- geodesics of M are also a geodesic of $\overline{M}$

<u>proof</u> (see notes for a "synthetic" proof based on Thales thm)

We will use a modification of the pf. of Lemma 4.19.

<u>Observe</u> by Lem 4.19 $s \to d(c(s), c(\bar{s}))$ stays bounded $\Rightarrow$ it must be constant (because it is a convex function defined in the whole $\mathbb{R}$)

Introduce coordinates
$$F : [0,1] \times \mathbb{R} \longrightarrow M$$
$$(s,t) \longmapsto F(s,t)$$



$$V_{s_0} = F_* \frac{\partial}{\partial s} \qquad \gamma'_{s_0} = F_* \frac{\partial}{\partial t}$$

<u>Recall</u>

$$0 = \frac{d^2}{ds^2}\Big|_{s=0} L(\gamma_{s_0+s}) = \int_0^1 \overbrace{|(V'_{s_0})^\perp|^2}^{\geq 0} - \overbrace{R(V_{s_0},\gamma'_{s_0},V_{s_0},\gamma'_{s_0})}^{\geq 0}\, dt$$

$$+ \underbrace{\text{" end point terms "}}_{=0}$$

(i) $R\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}, \frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) \equiv 0$

image(F) $\subset M$ has 0 Gauss curvature

(ii)

g_{st} metric in (s,t) coordinates of $F([0,1]\times\mathbb{R})$

$= \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ = constant matrix !

(iii) $\frac{\bar{D}}{\partial t} F_* \frac{\partial}{\partial s} = 0 \qquad \frac{\bar{D}}{\partial s} F_* \frac{\partial}{\partial s} = 0$

$\bar{D}$ is the Levi Civita of M,

D Levi Civita of image(F)

$\Rightarrow \bar{D} - D \equiv 0$ on image(F)

$\Leftarrow$ 2nd FF of $\mathrm{imag}(F) \equiv 0$ $\blacksquare$

Isometries of Hadamard mfld's

<u>Defin</u> 4.21 M Hadamard mfld, $\gamma \in \mathrm{Isom}(M)$

<u>displacement</u> $d_\gamma : M \longrightarrow [0, \infty)$

$$d_\gamma(p) = d(p, \gamma(p))$$

$$|\gamma| := \inf\{ d_\gamma(p) : p \in M \}$$

$$\mathrm{Min}(\gamma) := \{ p \in M : d_\gamma(p) = |\gamma| \}$$

Lemma 4.22 $A := \mathrm{Min}(\gamma)$ is closed, $\gamma(A) = A$,

A convex (i.e. geodesically convex)

---

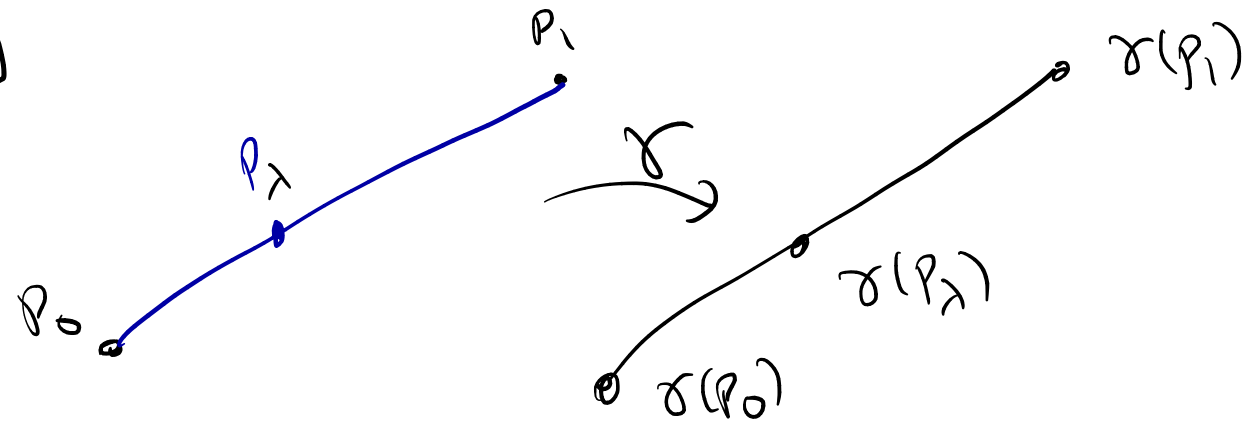
proof
- A is closed because $d : M \times M \to \mathbb{R}_+$ and γ are continuous
- Fix $p \in A$, $d_\gamma(\gamma(p)) = d(\gamma(p), \gamma^2(p)) = d(p, \gamma(p)) = |\gamma|$

  $(\Rightarrow \gamma(p) \in A)$

- It remains to show convexity. Fix $p_0, p_1 \in A$

p_λ, $\lambda \in [0,1]$

traces the geodesic segment joining p_0 & p_1



Using Lemma 4.19

$$|\gamma| \leq d(p_\lambda, \gamma(p_\lambda)) \leq (1-\lambda)\, d(p_0, \gamma(p_0)) + \lambda\, d(p_1, \gamma(p_1))$$
$$= (1-\lambda)|\gamma| + \lambda|\gamma| = |\gamma|$$

$$\Longrightarrow \quad p_\lambda \in A$$

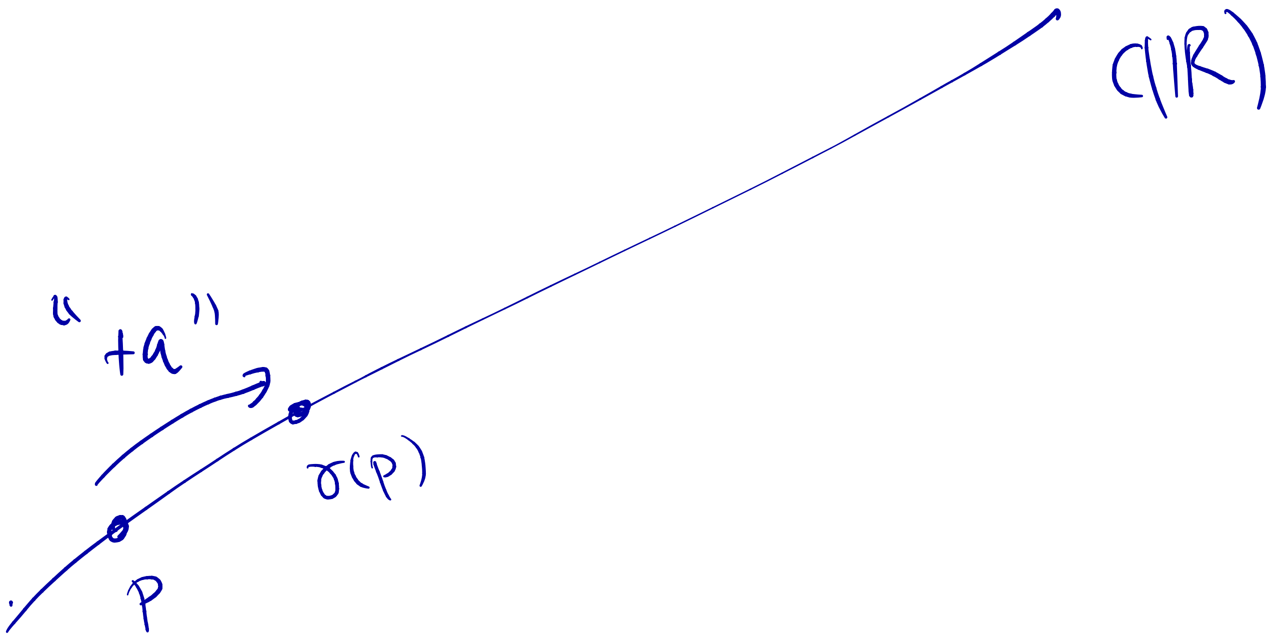
Def'n 4.23 Isometry γ is called

- parabolic if $\mathrm{Min}(\gamma) = \emptyset$
- elliptic if $|\gamma| = 0$
- hyperbolic if $|\gamma| > 0$

$\Big\}$ and no parabolic

If γ is isometry, unit speed geodesic $c : \mathbb{R} \to M$ is called <u>axis</u> if $\exists\, a > 0$ s.t.

$$\gamma(c(s)) = c(s+a) \quad (\forall s \in \mathbb{R})$$

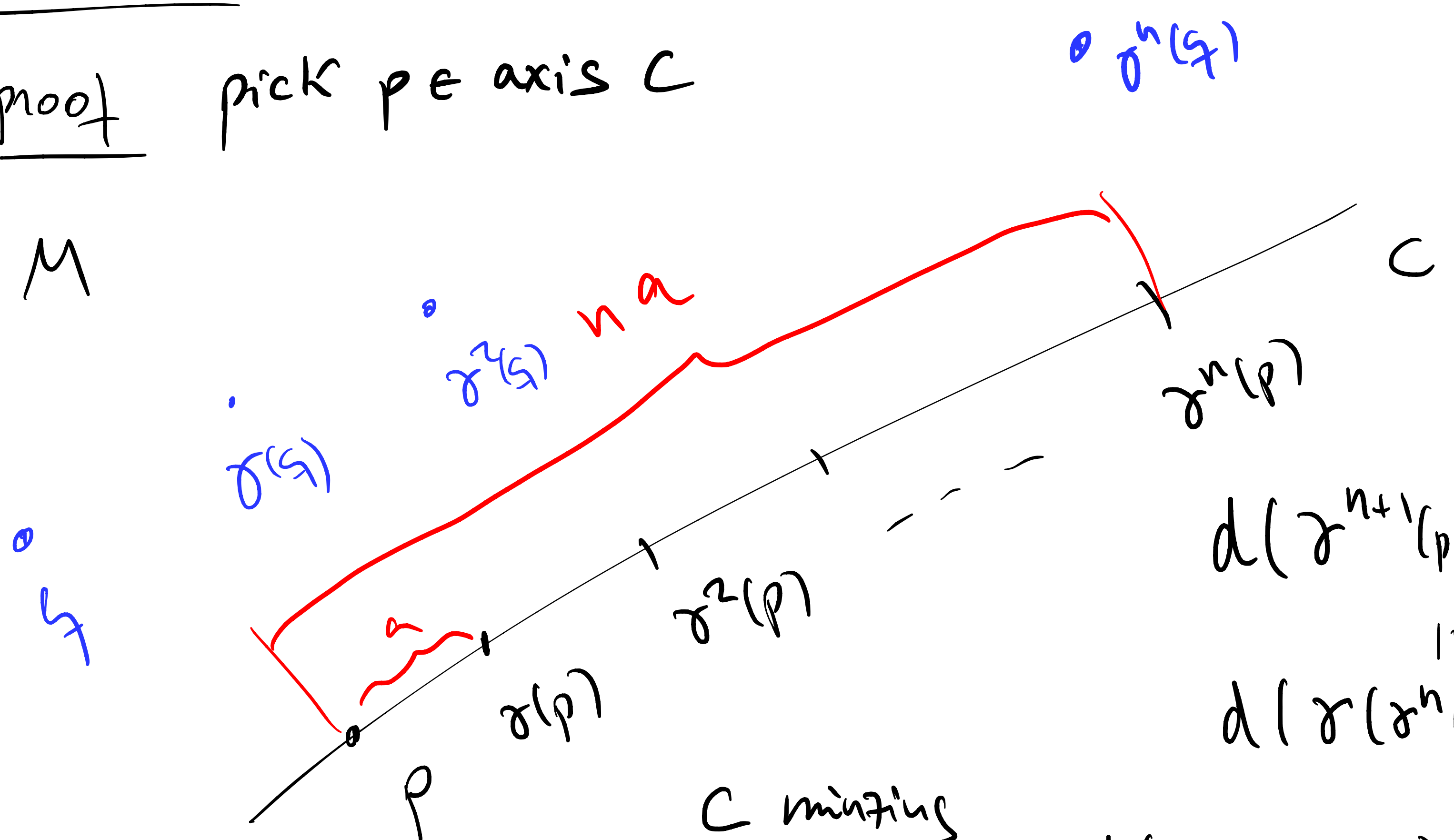


Lemma 4.24 $\gamma \in \mathrm{Isom}(M)$, M Hadamard,

- γ has axis c $\Longrightarrow$ $a = |\gamma|$
- $|\gamma| > 0$ $\Longrightarrow$ γ has an axis

---

proof pick $p \in$ axis c



$$d(\gamma^{n+1}(p), \gamma^n(p)) = a$$

$$\|$$

$$d(\gamma(\gamma^n(p)), \gamma^n(p))$$

c minimizing $\Longrightarrow$ $d(\gamma^n(p), p) = na$

Pick $q \in M$

$$d(\gamma^n(q), \gamma^n(p)) = d(q,p) \qquad (\gamma \text{ isom.})$$

By triangle ineq. :

$$na = d(p, \gamma^n(p)) \leq d(p,q) + \sum_{i=0}^{n-1} d(\gamma^i(q), \gamma^{i+1}(q)) + d(\gamma^n(q), \gamma^n(p))$$

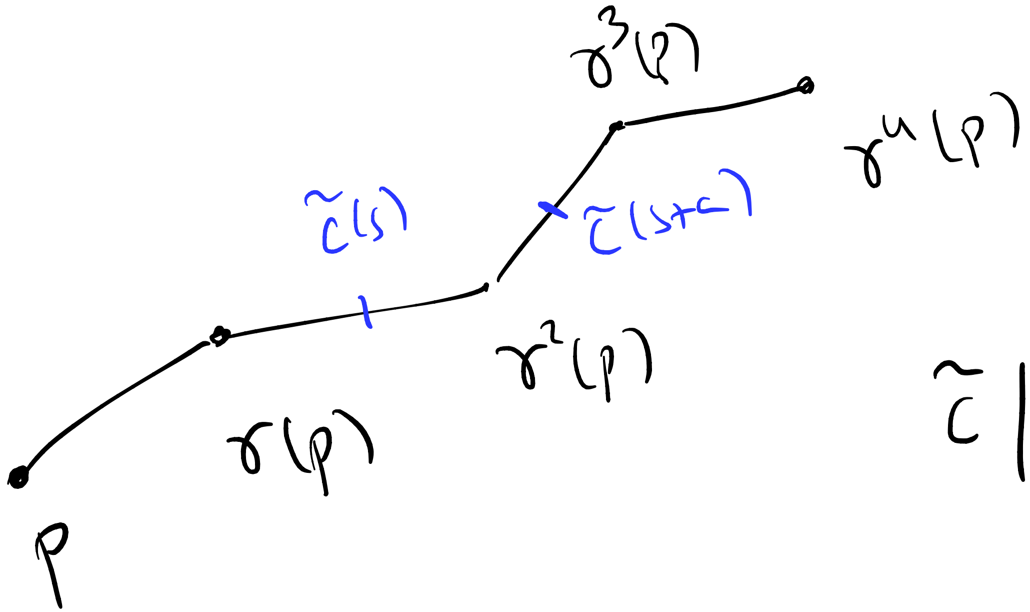
$$= 2\,d(p,q) + n\, d(\gamma(q), q)$$

Send $n \nearrow \infty$

$$d(\gamma(p), p) = a \leq d(\gamma(q), q) \Rightarrow \underline{a = |\gamma|}$$

suppose inf is attained $\Rightarrow$ $\exists p \in M$ s.t

$$a := |\gamma| = d(p, \gamma(p)) \leq d(q, \gamma(q)) \quad (\forall q \in M)$$



$\tilde{c}|_{[0,a]}$ unit speed geodesic segment joining p and $\gamma(p)$

$$\tilde{c}(na+t) = \gamma^n(\tilde{c}(t))$$

$$a = L(\tilde{c}|_{[s,s+a]}) \overset{(*)}{\geq} d(\tilde{c}(s), \tilde{c}(s+a))$$

$$= d(\tilde{c}(s), \gamma(\tilde{c}(s)))$$

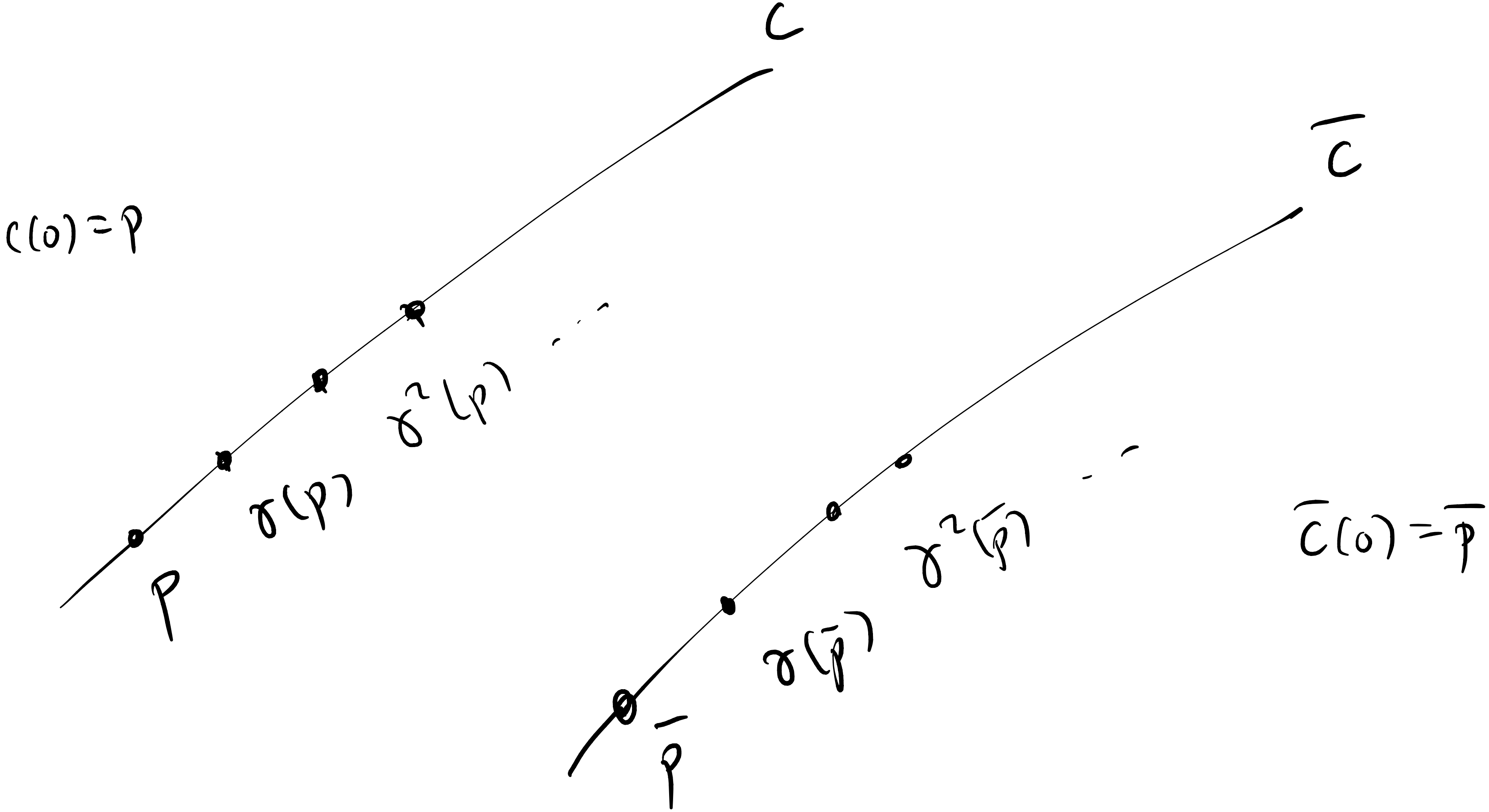
$$\geq |\gamma| = a$$

$\Rightarrow$ (*) holds with $=$ $\Rightarrow$ no angles

$\Rightarrow$ $\tilde{c}$ is a geodesic and hence an axis

$\boxtimes$

<u>Observation</u> If γ has 2 different axis



$$d(c(na), \bar{c}(na)) = d(p, \bar{p}) \quad \forall n \in \mathbb{Z}$$

$\Rightarrow$ $d(c(t), \bar{c}(t))$ stays bdd $\forall t \in \mathbb{R}$

and we can apply Prop 1.20!

<u>Thm 4.26</u> (Preismann 1942) If M is <u>closed</u> (compact, in part complete) with $\sec < 0$, then every abelian subgroup of $\pi_1(M)$ is isomorphic to $\mathbb{Z}$.

<u>proof</u> By Hadamard-Cartan, calling $\bar{M}$:= universal cover of M, $(\pi : \bar{M} \longrightarrow M$ Riem. covering map$)$

Γ := group of deck tr. $\cong$ $\pi_1(M)$

Fix $\gamma \in \Gamma$. Since M is compact $|\gamma|$ is attained

$$\inf_{p \in \overline{M}} d(\gamma(p), p) = \min_{q \in M} d(\gamma(\pi^{-1}(q)), \pi^{-1}(q))$$

Also, γ being deck tr. $|\gamma| > 0$

$\overset{\text{Lem. 4.24}}{\Longrightarrow}$ γ has a <u>unique</u> (by Prop 4.20, $\overset{\text{using}}{\sec < 0}$) <u>axis</u>

Let us call this axis $L_\gamma \subset \overline{M}$

Take $\beta \in \Gamma$ be some isometry that commutes with γ

(i.e., $\beta \circ \gamma = \gamma \circ \beta$)

$$\gamma(\beta L_\gamma) = \beta(\gamma L_\gamma) = (\beta L_\gamma) \implies \beta L_\gamma \text{ is an axis of } \gamma$$

$$\implies \beta L_\gamma = L_\gamma$$

$$\implies L_\beta = L_\gamma =: L$$

independent of β

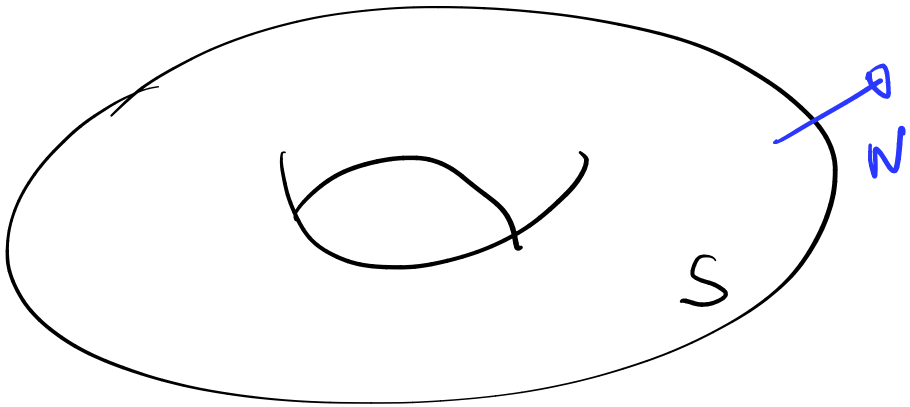
Therefore, any abelian subgroup of Γ acts by translation a single line $L \cong \mathbb{R}$ (without fixed pts!) (i.e., $\gamma(x) = x + a$) $\implies$ it must be isomorphic to $\mathbb{Z}$ ■

First variation of area

"Model problem" M is a 3-manifold (e.g. $\mathbb{R}^3$)

$S \subset M$ is 2-dim. submanifold "$S_\varepsilon := S + \varepsilon N \xi$"

$\xi \in C^\infty(S)$



$x \longmapsto x + \varepsilon N(x)\xi(x)$

parametrization of S_ε

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} A(S_\varepsilon) = ??$$

More general setting | $\dim(M) = m$, $n = m-1$

$$f : \underset{\subset \mathbb{R}^n}{U} \times (-\varepsilon_0, \varepsilon_0) \longrightarrow M$$

$\left(f(x) := \exp_{\psi(x)}(\varepsilon N(\psi(x))) \right.$

$\psi : \underset{\subset \mathbb{R}^n}{U} \longrightarrow S$ local param.

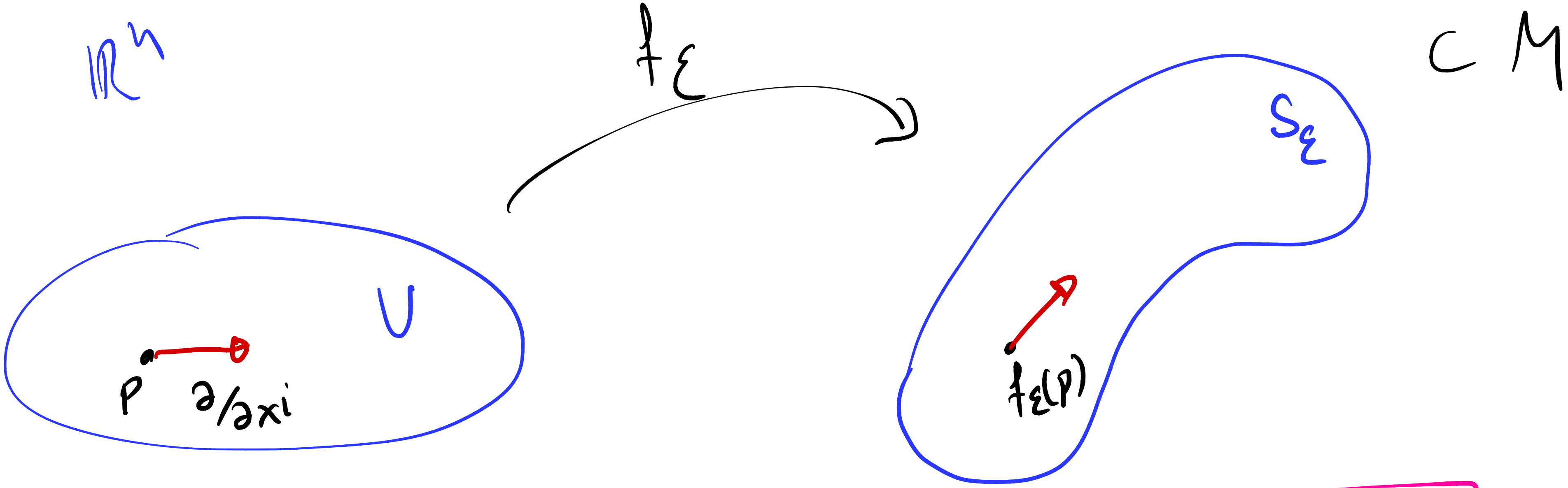
$\left. \text{for some } N \in \Gamma(TS^\perp) \right)$

$$f_\varepsilon : f(\cdot, \varepsilon) \,, \quad \boxed{S_\varepsilon = f_\varepsilon(U)}$$

Assume in addition $f_\varepsilon(p) = f_0(p) \quad \forall p \in \partial U$

$$A(S_\varepsilon) = \int_U \sqrt{\det g_{\varepsilon,ij}} \, dx$$

For $x \in U$, $\quad g_{\varepsilon,ij}(x) := g\left(f_{\varepsilon *} \frac{\partial}{\partial x^i}, f_{\varepsilon *} \frac{\partial}{\partial x^j} \right)$



Assume for simplicity $\boxed{f_* \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \text{ perpendicular to } S = S_0}$ (#)

We want to compute

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} A(S_\varepsilon) = \int_U \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \sqrt{\det g_{\varepsilon,ij}}\, dx$$

Let us show first:

$$\frac{1}{2}\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} g_{\varepsilon,ij} \overset{☺}{=} -\left\langle h_{ij},\ f_*\frac{\partial}{\partial\varepsilon}\Big|_{\varepsilon=0}\right\rangle$$

where $h_{ij}(x) := h\left(f_{0*}\frac{\partial}{\partial x^i},\ f_{0*}\frac{\partial}{\partial x^j}\right)$

2nd f.f

$$\boxed{\begin{array}{ll} h(X,Y) = \overline{D}_X Y - D_X Y & \overline{D} \text{ cov. dif of } M \\ & D \text{ cov dif of } S_0 \\ X,Y\in\Gamma(TS_0)\subset\Gamma(TM) & \end{array}}$$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} g_{\varepsilon,ij} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left\langle f_*\frac{\partial}{\partial x^i},\ f_*\frac{\partial}{\partial x^j}\right\rangle_M$$

$$= \underbrace{\left\langle \frac{\overline{D}}{\partial\varepsilon} f_*\frac{\partial}{\partial x^i},\ f_*\frac{\partial}{\partial x^j}\right\rangle\Big|_{\varepsilon=0}}_{(\mathrm{I})} + \underbrace{\left\langle f_*\frac{\partial}{\partial x^i},\ \frac{\overline{D}}{\partial\varepsilon} f_*\frac{\partial}{\partial x^j}\right\rangle\Big|_{\varepsilon=0}}_{(\mathrm{II})}$$

$$(\mathrm{I}) = \left\langle \frac{\bar{D}}{\partial x^i} f_* \frac{\partial}{\partial \varepsilon},\ f_* \frac{\partial}{\partial x^j} \right\rangle \Big|_{\varepsilon=0}$$

$$= \frac{\partial}{\partial x^i}\left(\underbrace{\left\langle f_* \frac{\partial}{\partial \varepsilon},\ f_* \frac{\partial}{\partial x^j} \right\rangle \Big|_{\varepsilon=0}}_{\equiv 0 \ (\#)} \right) - \left\langle f_* \frac{\partial}{\partial \varepsilon},\ \frac{\bar{D}}{\partial x^i} f_* \frac{\partial}{\partial x^j} \right\rangle \Big|_{\varepsilon=0}$$

$$= - \left\langle f_* \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0},\ \left(\frac{\bar{D}}{\partial x^i} f_* \frac{\partial}{\partial x^j} \right)^{\perp} \right\rangle$$

$$\left(\frac{\bar{D}}{\partial x^i} f_* \frac{\partial}{\partial x^j} \right)^{\perp} = \left(\frac{\bar{D}}{\partial x^i} - \frac{D}{\partial x^i} \right)\left(f_* \frac{\partial}{\partial x^j} \right)$$

defn h

$$= - \left\langle f_* \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0},\ h_{ij} \right\rangle$$

$$\left[\begin{array}{l}\text{Differentiation of det. Recall } A(t) \text{ matrix curve with } A(0) = \mathrm{Id} \\ \left.\frac{d}{dt}\right|_{t=0} \det(A(t)) = \mathrm{tr}\left(\left.\frac{d}{dt}\right|_{t=0} A(t)\right) \qquad \underset{\shortparallel}{\sqrt{\det(g_0^{-1} g_\varepsilon)}} \sqrt{\det g_0}\end{array}\right]$$

We differentiate in ε : $\quad A(S_\varepsilon) = \int_U \sqrt{\det g_{\varepsilon,ij}}\, dx$

$$\boxed{g = g_0}$$

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} A(S_\varepsilon) = \int_U \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \sqrt{\det(g^{ki} g_{\varepsilon,ij})} \sqrt{\det g_{ik}}\, dx$$

$$\boxed{\sqrt{g}(x) := \sqrt{\det g_{ij}(x)}}$$

$$= \int_U \frac{\delta^j_k\, g^{ki} \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} g_{\varepsilon,ij}}{2\sqrt{1}} \sqrt{g}\, dx$$

$$= \int_U -\left\langle g^{ij} h_{ij},\ f_* \left.\frac{\partial}{\partial\varepsilon}\right|_{\varepsilon=0} \right\rangle \underbrace{\sqrt{g}\, dx}_{d\mathrm{Vol}\ \ S = S_0}$$

$$= \int_V -\langle \vec{H}, f_* \frac{\partial}{\partial \varepsilon}\Big|_{\varepsilon=0} \rangle$$

$$n\vec{H}(x) = g^{ij} h_{ij}(x) \in \Gamma(TS^\perp)$$

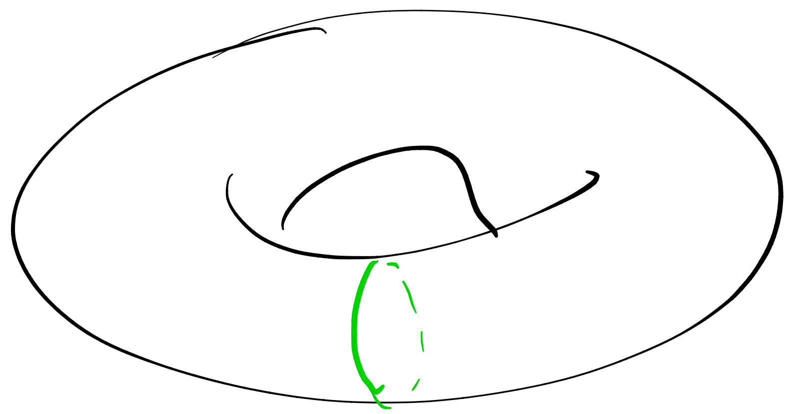
is called vectorial mean curvature

<u>Applications of 1st variation formula</u>

M complete Riem. mfold of dim m

S is n-dim complete submanifold $\quad n = m-1$

S is <u>minimal</u> if



for all variations

$$F_\varepsilon(p) = \exp_p(\varepsilon N(p)),$$

where $N \in \Gamma(TS^\perp)$ with cpt. supp.

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} A(S_\varepsilon) = 0$$

"i.e. if it is a <u>critical pt. of Area</u>"

<u>Exercise 1</u> Show that S is minimal

$$\Longleftrightarrow \quad \vec{H} \equiv 0 \text{ on } S$$

Exercise 2 If S is enclosing some volume V and we S has minimal area among (smooth) all surfaces enclosing volume V, then:

$$\langle \vec{H}, \nu \rangle \equiv ct \text{ on } S$$ (if S is connected)

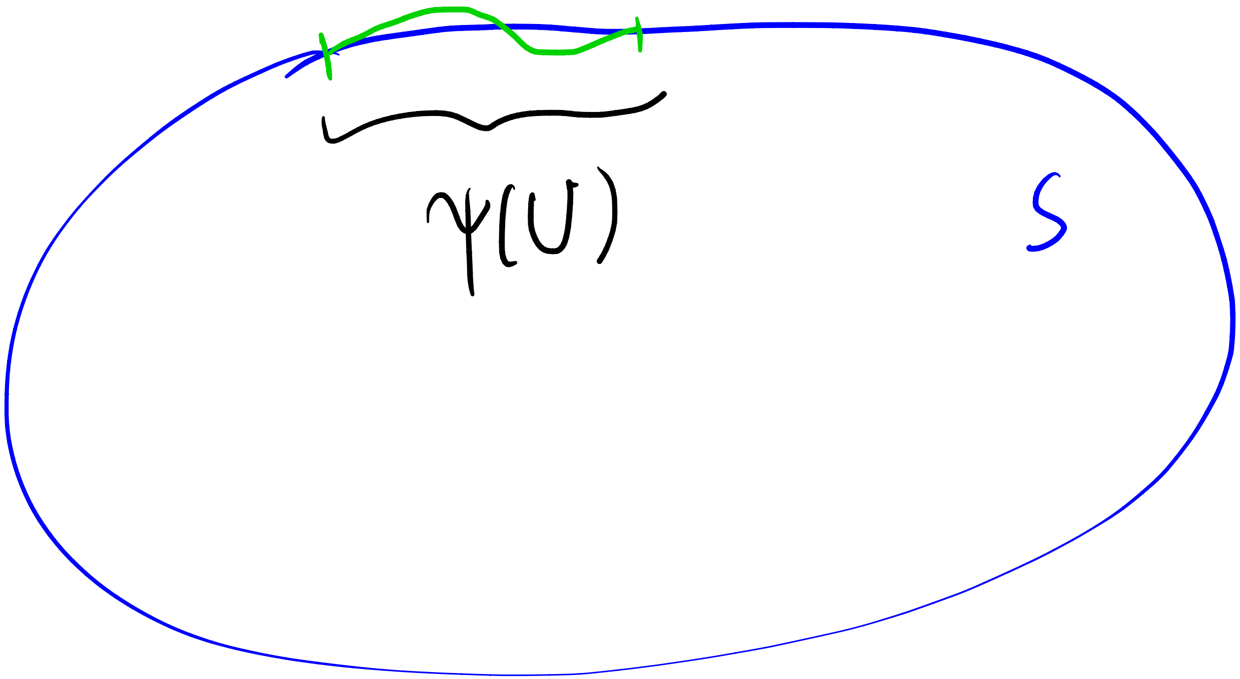
ν is a unit normal vector field (to S)

Hint: Chose any local param $\psi : U^{\subset \mathbb{R}^n} \longrightarrow S$ and $u : U \to \mathbb{R}$ (smooth, bdd, cpt supported)

$N = u \circ \psi^{-1} \cdot \nu$

$$f_\varepsilon(x) = F_\varepsilon \circ \psi(x) = \exp_{\psi(x)}\left(u(x)\, \nu(\psi(x))\right)$$

the volume enclosed by the S_ε is the same as the one for S_0 $\left(\pm O(\varepsilon^2)\right)$ iff $\int_U u\sqrt{g}\,dx = 0$



2^{nd} Variation of Area

$\langle\ ,\ \rangle$: Riem. metric of ambient mfld M

$\bar{D}$: Levy Civita of M

The setting: $\boxed{n = m-1}$

$$f : \underset{\subset \mathbb{R}^n}{U} \times (-\varepsilon_0, \varepsilon_0) \longrightarrow M^m$$

(assume f_0 is immersion)

$$f_\varepsilon := f(\cdot, \varepsilon), \quad S_\varepsilon := f_\varepsilon(U)$$

$$\partial_i = \begin{cases} f_* \dfrac{\partial}{\partial x^i} \\ \dfrac{\partial}{\partial x_i} \end{cases}, \qquad \partial_\varepsilon = \begin{cases} f_* \partial_\varepsilon \\ \partial_\varepsilon \end{cases}$$

$$g_{\varepsilon ij} = \langle \partial_i, \partial_j \rangle := \left\langle f_* \frac{\partial}{\partial x_i}, f_* \frac{\partial}{\partial x_j} \right\rangle$$

Goal computation applies to $f(x) = \exp_{\varphi(x)}(\varepsilon N(\varphi(x)))$

$$\partial_\varepsilon \perp S_0 \,, \qquad \bar{D}_{\partial_\varepsilon} \partial_\varepsilon \equiv 0$$

$$\left[\frac{d}{dh}\Big|_{h=0} \det(g_\varepsilon^{-1} g_{\varepsilon+h}) \det(g_\varepsilon) \right]$$

Key computation

$$\frac{d}{d\varepsilon} \sqrt{g_\varepsilon} = \frac{1}{2} g_\varepsilon^{ij} \frac{d}{d\varepsilon} g_{\varepsilon,ij} \sqrt{g_\varepsilon} \qquad (\star)$$

$$\frac{d}{d\varepsilon} g_{\varepsilon,ij} = \partial_\varepsilon \langle \partial_i, \partial_j \rangle \overset{\text{compatible}}{=} \langle \bar{D}_{\partial_\varepsilon} \partial_i, \partial_j \rangle + \langle \partial_i, \bar{D}_{\partial_\varepsilon} \partial_j \rangle$$

$$\overset{\text{torsion free}}{=} \langle \bar{D}_{\partial_i} \partial_\varepsilon, \partial_j \rangle + \langle \partial_i, \bar{D}_{\partial_j} \partial_\varepsilon \rangle$$

(☾) Vanishes at $\varepsilon = 0$!

$$\frac{d}{d\varepsilon} g_{\varepsilon,ij} = \boxed{\partial_i \langle \partial_\varepsilon, \partial_j \rangle + \partial_j \langle \partial_\varepsilon, \partial_i \rangle} - 2 \langle \partial_\varepsilon, \bar{D}_{\partial_i} \partial_j \rangle$$

Next step: Compute $\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \sqrt{g_\varepsilon}$

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} \sqrt{g_\varepsilon} \overset{(*)}{=} \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Big(\underbrace{g_\varepsilon^{ij}}_{1} \underbrace{\frac{1}{2}\frac{d}{d\varepsilon} g_{\varepsilon ij}}_{2} \underbrace{\sqrt{g_\varepsilon}}_{3} \Big)$$

$' := \frac{d}{d\varepsilon}$

1 $\quad g_\varepsilon^{ik} g_{\varepsilon kj} = \delta^i_j \implies (g^{ik})' g_{kj} + g^{ik} (g_{kj})' = 0$

$$\implies (g^{ik})' \underbrace{g_{kj} g^{jl}}_{\delta_k^l} + g^{ik} g^{jl} (g_{kj})' = 0$$

$$\implies (g^{il})' = -g^{ik} g^{jl} (g_{kj})'$$

$$(g^{ij})'\Big|_{\varepsilon=0} = -g^{ik}g^{jl}(g_{kl})'\Big|_{\varepsilon=0} = 2g^{ik}g^{jl}\langle\partial_\varepsilon, \overline{D}_{\partial_k}\partial_l\rangle$$

② $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}(C)$ $\qquad$ [Recall $\overline{D}_{\partial_\varepsilon}\partial_\varepsilon \equiv 0$]

$$\frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0}\frac{1}{2}g_{\varepsilon,ij} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(\frac{1}{2}\{\partial_i\langle\partial_\varepsilon,\partial_j\rangle + \partial_j\langle\partial_\varepsilon,\partial_i\rangle\} - \langle\partial_\varepsilon, \overline{D}_{\partial_i}\partial_j\rangle\Big)$$

$$= \frac{1}{2}\Big(\partial_\varepsilon\partial_i\langle\partial_\varepsilon,\partial_j\rangle + \partial_\varepsilon\partial_j\langle\partial_\varepsilon,\partial_i\rangle\Big) - \partial_\varepsilon\langle\partial_\varepsilon, \overline{D}_{\partial_i}\partial_j\rangle$$

$$= \frac{1}{2}\Big(2\langle\overline{D}_{\partial_i}\partial_\varepsilon, \overline{D}_{\partial_j}\partial_\varepsilon\rangle + \underbrace{\langle\partial_\varepsilon, \overline{D}_i\overline{D}_j\partial_\varepsilon\rangle}_{} + \underbrace{\langle\partial_\varepsilon, \overline{D}_j\overline{D}_i\partial_\varepsilon\rangle}_{}\Big)$$

$=$

$$\underbrace{- \langle \partial_\varepsilon , \bar{D}_\varepsilon \bar{D}_i \partial_j \rangle}$$

$$\left(\begin{aligned} \bar{D}_\varepsilon \bar{D}_i \partial_j &= \bar{D}_i \bar{D}_\varepsilon \partial_j + \bar{R}(\partial_\varepsilon, \partial_i)\partial_j \\ &= \underbrace{\bar{D}_i \bar{D}_j \partial_\varepsilon} + \bar{R}(\partial_\varepsilon, \partial_j)\partial_j \end{aligned} \right)$$

$$\ldots = \langle \bar{D}_i \partial_\varepsilon , \bar{D}_j \partial_\varepsilon \rangle - \underbrace{\langle \overset{V}{\partial_\varepsilon}, \bar{R}(\overset{X}{\partial_\varepsilon}, \overset{Y}{\partial_j}) \overset{W}{\partial_j} \rangle}_{\bar{R}(\partial_\varepsilon, \partial_j, \partial_\varepsilon, \partial_i)}$$

Hence,

$$g^{ij} \frac{1}{2} \frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} g_{\varepsilon,ij} = g^{ij} \langle \bar{D}_i \partial_\varepsilon , \bar{D}_j \partial_\varepsilon \rangle \Big|_{\varepsilon=0} - \overline{Ric}(\partial_\varepsilon, \partial_\varepsilon)$$

3 $(\star) + (\text{☾}) \big|_{\varepsilon=0}$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \sqrt{g_\varepsilon} = -g^{ij}\langle h_{ij}, \partial_\varepsilon\rangle\Big|_{\varepsilon=0} \sqrt{|g|}$$

Putting everything together:

$$\frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} \sqrt{g_\varepsilon} = -2g^{ik}g^{jl}\langle h_{kl}, \partial_\varepsilon\rangle\langle h_{ij}, \partial_\varepsilon\rangle\sqrt{g} \qquad (1)$$

$$+\left(g^{ij}\langle \bar{D}_i\partial_\varepsilon, \bar{D}_j\partial_\varepsilon\rangle - \overline{Ric}(\partial_\varepsilon, \partial_\varepsilon)\right)\sqrt{g} \qquad (2)$$

$$+ \quad \langle g^{ij}h_{ij}, \partial_\varepsilon\rangle^2 \sqrt{g} \qquad (3)$$

<u>Using</u> $n = m-1$ (co-dim 1)

Fix ν <u>unit</u> normal vector to S_0

$A := \langle h, \nu \rangle$ symmetric 2-tensor on S_0

$A_{ij} := \langle h_{ij}, \nu \rangle$ (actually $\nu \circ \Psi$) on U

$H := \langle g^{ij} h_{ij}, \nu \rangle$ scalar mean curv.

<u>Recall</u>

$N = u\nu$

$f(x) = \exp_{\Psi(x)}(\varepsilon N(\Psi(x)))$

$u : S_0 \longrightarrow \mathbb{R}$

$$\boxed{\partial_\varepsilon = u\nu}$$

$$2 g^{ik} g^{jl} \langle h_{kl}, \partial_\xi \rangle \langle h_{ij}, \partial_\xi \rangle = 2u^2 \, g^{ik} g^{jl} A_{kl} A_{ij}$$

(actually $u \circ \psi$)

$$= 2u^2 \sum_{i,j=1}^{n} A(e_i, e_j) A(e_i, e_j)$$

$\boxed{e_i \text{ ONB}}$

$$(1) = -\underbrace{2u^2 |A|^2}\sqrt{g}$$

Hilbert Schmidt norm

$$|A|^2 = \sum_{i,j=1}^{n} A_{ij}^2 = \sum_{i=1}^{n} \lambda_i^2$$

this is coordinate free!

(λ_i eigenv. of A)

since $\overline{D_i}\, \partial_\xi = \overline{D_i}(u\nu) = \partial_i u\, \nu + u\, \overline{D_i}\, \nu$:

$$(2) = \Big(\underbrace{g^{ij} \partial_i u\, \partial_j u}_{\langle \operatorname{grad} u, \operatorname{grad} u \rangle} + u^2 \underbrace{g^{ij} \langle \overline{D_i} \nu, \overline{D_j} \nu \rangle}_{g^{ik} g^{jl} A_{kl} A_{ij} \text{ (exercise*)}} - u^2 \overline{\operatorname{Ric}}(\nu, \nu)\Big)\sqrt{g}$$

*Hint $\langle \overline{D}_i \nu, \partial_j \rangle = - \langle \nu, \overline{D}_i \partial_j \rangle$

$$③ = H^2 \eta^2 \sqrt{g}$$

So, summarising, we proved

$$\frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon=0} \int_U \sqrt{g_\varepsilon}\, dx = \int_U \frac{d^2}{d^2\varepsilon}\bigg|_{\varepsilon=0} \sqrt{g_\varepsilon}\, dx$$

$$= \int_U \Big\{ \underbrace{g^{ij} \partial_i u \, \partial_j u}_{|\operatorname{grad} u|^2} + \left(H^2 - |A|^2 - \overline{Ric}(\nu,\nu) \right) \eta^2 \Big\} \underbrace{\sqrt{g}\, dx}_{d\mathrm{Vol}}$$

Therefore, we have proved :

Thm Given $S_0 \subset M$ smooth embeded submfld of codim 1,
For given $u: S_0 \longrightarrow \mathbb{R}$ (smooth) consider the variation

$$F_\varepsilon(p) = \exp_p(\varepsilon u \nu(p)) \quad \Big| \quad S_\varepsilon := F_\varepsilon(S_0)$$

$$\frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} A(S_\varepsilon) = \int_{S_0} |\mathrm{grad}\, u|^2 + \left(H^2 - |A|^2 - \mathrm{Ric}(\nu,\nu)\right) u^2 \, dV$$

Def a min surf. is called **stable** if it has ≥ 0 2nd variation

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Cor 1 There are **no** stable closed minimal (hyper)surfaces on a positively curved manifold M

proof Choose $u = 1$

$$\int_S -|A|^2 - Ric(\nu,\nu) \, dV \geq 0$$

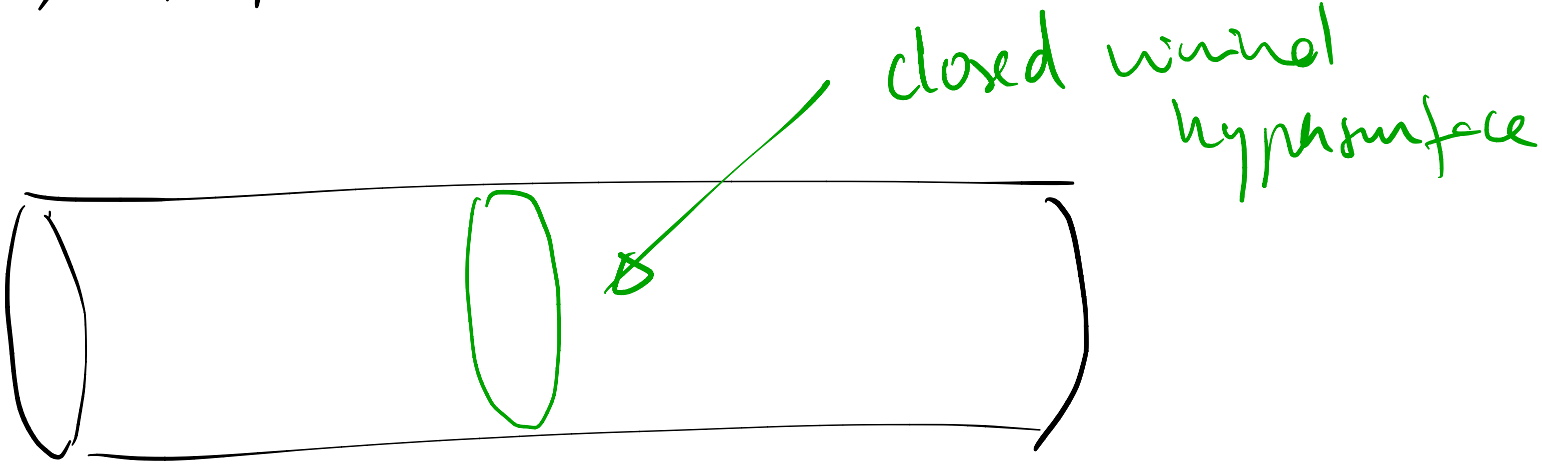
Cor 2 On mfld with $Ric \geq 0$, every closed, stable, min hypersurface must be totally geodesic

proof Similarly as before, we get

Stability $\Rightarrow$ $|A|^2 \equiv 0$

Example

Cylinders



__Conjecture__ In the Euclidean space $\mathbb{R}^n$ every complete, stable, embedded min submanifold must be a hyperplane, if $n < 8$

- $n = 3, 4$ ✓
- $n \geq 8$ Simons cone counterexample

Thm (Simon–Yau) If M^3 is a 3-dim mfld with >0 scalar curvature, and S is an orientable, connected, closed stable min. surface in M. Then S is diffeomorphic to S^2.

Exercise Use Gauss eq'ns to show $S^2 \subset M^3$

$$(*)\quad sc_S = sc_M - 2\,\mathrm{Ric}(\nu,\nu) + H^2 - |A|^2$$

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pf. of SY thm. S is stable minimal $(H \equiv 0)$

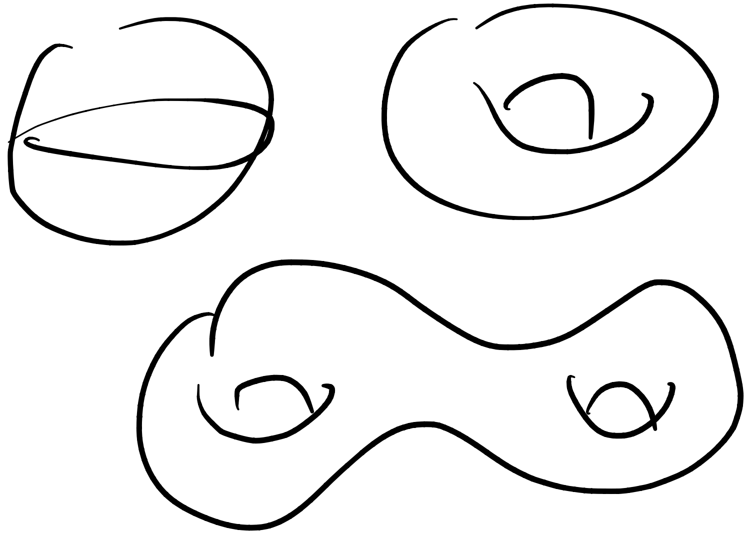
$$\Rightarrow \int_S -|A|^2 - \mathrm{Ric}(\nu,\nu)\, d\mathrm{Vol} \geq 0$$

$$\Leftrightarrow \int_S -\frac{1}{2}|A|^2 + \frac{1}{2}\Big(K_S - \underbrace{sc_M}_{>0}\Big) \geq 0$$

$$\Rightarrow \int_S K_S > 0$$

(S closed orientable 2-dim mfld)

Using Gauss-Bonnet



$\Rightarrow$ the genus of the surface must be 0

$\Rightarrow$ S is diffeomorphic to S^2