Exercise Sheet 1

1. Divergence Theorem

Let $M \subset \mathbb{R}^3$ be a compact 3-dimensional manifold with boundary, $N \colon \partial M \to S^2$ the outward pointing unit normal,

$$\pi = f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$$

a 2-form on $\mathbb{R}^3$ and $X = (f, g, h)$.

(a) Show that $d\pi = \operatorname{div}(X) dx \wedge dy \wedge dz$.

(b) Deduce the Divergence Theorem

$$\int_M \operatorname{div}(X)\, \mathrm{dVol} = \int_{\partial M} \langle X, N \rangle\, \mathrm{d}A$$

from the Theorem of Stokes for differential forms.

2. Orthogonal Structures

Let $\pi \colon E \to M$ be a vector bundle of rank k over a manifold M. An *orthogonal structure* g on E assigns to every point $p \in M$ a scalar product g_p on the fiber $E_p := \pi^{-1}(p)$, such that for all sections s, s' the map $p \mapsto g_p(s(p), s'(p))$ is smooth.

Prove that every vector bundle admits an orthogonal structure.

Hint: Use a partition of unity.

3. Vector Bundles of Rank 1

a) Prove that every vector bundle of rank 1 over a simply connected manifold is trivial.

b) Prove that, up to isomorphism, there exist exactly two vector bundles of rank 1 over S^1.