Exercise Sheet 10

1. Two dimensional Hadamard manifolds

Let (M, g) be a two dimensional Hadamard manifold. For fixed point $p \in M$ and isometry $H : \mathbb{R}^2 \to TM_p$, consider $(\mathbb{R}^2, \overline{g})$ where $\overline{g} := (\exp_p \circ H)^* g$.

(a) Show that $\overline{g}$ is of the form

$$\overline{g}_x(v, w) := \left(v \cdot \tfrac{x}{|x|}\right)\left(w \cdot \tfrac{x}{|x|}\right) + \frac{f^2(x)}{|x|^2}\left(v \cdot w - \left(v \cdot \tfrac{x}{|x|}\right)\left(w \cdot \tfrac{x}{|x|}\right)\right), \quad (1)$$

where $f^2(x)/|x|^2$ is smooth (also at $x = 0$) and has limit 1 as $x \to 0$, and where $t \mapsto f(tx)$ is nonnegative and convex for any fixed $x \in \mathbb{R}^2 \setminus \{0\}$.

(b) Reciprocally, show that $\mathbb{R}^2$ endowed with any metric $\overline{g}$ satisfying the properties established in (a) —and such that $g_x(v, w)$ extends to a smooth metric across $x = 0$— gives a model of a Hadamard manifold (simply connected with nonpositive sectional curvature at all points).

2. Some consequences of non-positive sectional curvature

Let M be a Hadamard manifold. Prove the following:

(a) For each $p \in M$, the map $(\exp_p)^{-1} \colon M \to TM_p$ is 1-Lipschitz.

(b) For $p, x, y \in M$, it holds

$$d(p, x)^2 + d(p, y)^2 - 2d(p, x)d(p, y) \cos \gamma \leq d(x, y)^2,$$

where γ denotes the angle in p.

(c) Let m denote the midpoint of the geodesic xy in M and let $p \in M$. Then we have

$$d(p, m)^2 \leq \frac{d(p, x)^2 + d(p, y)^2}{2} - \frac{1}{4} d(x, y)^2.$$

Hint: Prove it first in the Euclidean plane. (This is a rather difficult but interesting to do exercise if one uses only the results up to Chapter 4 in Prof. Lang's notes. The exercise becomes simpler if one uses the content of Chapter 5.)

3. Isometries with bounded orbits.

Let M be a Hadamard manifold. Prove the following:

(a) If $Y \subset M$ is a bounded set, then there is a unique point $c_Y \in M$ such that $Y \subset \overline{B}(c_Y, r)$, where $r := \inf\{s > 0 : \exists x \in M \text{ such that } Y \subset \overline{B}(x, s)\}$.

Hint: Prove it first in Euclidean space. It may be useful to use part (c) of exercise 2. We call c_Y the *center* of Y.

(b) Let γ be an isometry of M. Then γ is elliptic if and only if M has a bounded orbit. Furthermore, if γ^n is elliptic for some integer $n \neq 0$, then γ is elliptic.