Exercise Sheet 11

1. Asymptotic expansion of the circumference

Let M be a manifold, $E \subset TM_p$ a linear 2-plane and $\gamma_r \subset E$ a circle with center 0 and radius $r > 0$ sufficiently small. Show that

$$L(\exp(\gamma_r)) = 2\pi \left(r - \frac{\sec(E)}{6} r^3 + \mathcal{O}(r^4) \right)$$

for $r \to 0$.

2. Isoperimetric problem in two dimensional Hadamard manifolds

Let M be a 2-dimensional Hadamard manifold. Given $\Omega \subset M$ bounded, we say that $\partial\Omega$ is C^2 if it consists of a finite disjoint union of C^2 simple close curves. For such Ω define the *isoperimetric quotient*

$$\mathcal{I}(\Omega) := \frac{\text{length}(\partial\Omega)}{\text{area}(\Omega)^{\frac{1}{2}}}.$$

a) Suppose first that M is isometric to the Euclidean plane. Show that if Ω_0 is a minimizer of $\mathcal{I}$ (such that $\partial\Omega_0$ is C^2) then

$$\mathcal{I}(\Omega_0) = \sqrt{4\pi} \quad \text{and } \Omega_0 \text{ is an Euclidean disc.}$$

Hint: Show that a smooth minimizer $\partial\Omega_0$ must consist of exactly simple curve γ, and prove (using the first variation of arc length) that the geodesic curvature κ_g of γ must be constant. Deduce that γ must trace a circle in $\mathbb{R}^2$.

b) In the case of nonnegative Gauss curvature $K \leq 0$, show that if Ω_0 is a minimizer of $\mathcal{I}$ (with $\partial\Omega_0$ of class C^2) then $\mathcal{I}(\Omega_0) = \sqrt{4\pi}$, and Ω_0 is isometric to an Euclidean ball.

Hint: Using small metric balls $B_r(p) \subset M$, with $r \ll 1$ as "competitors", prove that $\mathcal{I}(\Omega_0) \leq \sqrt{4\pi}$. Show that, as in a), $\partial\Omega_0$ must consist of only one closed simple curve γ. Let ν be the inwards unit normal to $\partial\Omega_0$, define (for ε small) $\gamma_\varepsilon(t) := \gamma(t) + \varepsilon\nu(t)$, and let Ω_ε be the bounded connected component of $M \setminus \text{image}(\gamma_\varepsilon)$. Show that $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} I(\Omega_\varepsilon) \leq 0$, and < 0 unless $K \equiv 0$ in Ω_0.

3. Characterization of the cut value

Let M be a complete Riemannian manifold. Given $p \in M$ and $u \in TM_p$ we define the *cut value* of u as the number

$$t_u := \sup\{t > 0 \ : \ d(\exp_p(tu), p) = t\}.$$

Let $c_u \colon \mathbb{R} \to M$, $c_u(t) := \exp_p(tu)$, be a unit speed geodesic. If the cut value t_u is finite then (at least) one of the following holds for $t = t_u$:

(i) $c_u(t)$ is the first conjugate point of p along $c_u|_{[0,t]}$,

(ii) there exists $v \in TM_p$, $|v| = 1$, $v \neq u$ with $c_u(t) = c_v(t)$.

Conversely, if (i) or (ii) is satisfied for some $t \in (0, \infty)$, then $t_u \leq t$.