Exercise Sheet 2

1. Basic properties of the Lie bracket

Let M be a smooth manifold, X , Y and Z belong to $\Gamma(TM)$, and f, g belong to $C^\infty(M)$

a) Show that the Lie bracket $[\cdot, \cdot]$ is bilinear and satisfies:

- $[Y, X] = -[X, Y]$
- $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$
- $[X, [Y, Z]] + [Y, [Z, X] + [Z, [X, Y]] = 0$

b) Show that in a chart (φ, U), if $X = \sum X^i \frac{\partial}{\partial \varphi^i}$ and $Y = \sum Y^j \frac{\partial}{\partial \varphi^j}$, we have

$$[X, Y]|_U = \sum_i \left(\sum_j X^j \frac{\partial Y^i}{\partial \varphi^j} - Y^j \frac{\partial X^i}{\partial \varphi^j} \right) \frac{\partial}{\partial \varphi^i}$$

2. The Levi-Civita connection on a submanifold

Let $(\bar{M}, \bar{g})$ be a Riemannian manifold with Levi-Civita connection $\bar{D}$, and let M be a submanifold of $\bar{M}$, equipped with the induced metric $g := i^*\bar{g}$, where $i\colon M \to \bar{M}$ is the inclusion map.

Show that the Levi-Civita connection D of (M, g) satisfies $D_X Y = (\bar{D}_X Y)^{\mathrm{T}}$ for all $X, Y \in \Gamma(TM)$, where the superscript T denotes the component tangential to M and $\bar{D}_X Y$ is defined(!) as $\bar{D}_X Y := \bar{D}_{\bar{X}} \bar{Y}$ for any extensions $\bar{X}, \bar{Y} \in \Gamma(T\bar{M})$ of X, Y.

3. Gradient and Hessian form

Let (M, g) be a Riemannian manifold, D the Levi-Civita connection and $f\colon M \to \mathbb{R}$ a smooth function on M.

a) The *gradient* $\mathrm{grad} f \in \Gamma(TM)$ is defined by

$$df(X) = g(\mathrm{grad} f, X), \quad \forall X \in \Gamma(TM).$$

Compute $\mathrm{grad} f$ in local coordinates.

b) The *Hessian form* $\mathrm{Hess}(f) \in \Gamma(T_{0,2}M)$ is defined by

$$\mathrm{Hess}(f)(X, Y) = g(D_X \mathrm{grad} f, Y), \quad \forall X, Y \in \Gamma(TM).$$

Prove that $\mathrm{Hess}(f)$ is symmetric and compute $\mathrm{Hess}(f)$ in local coordinates.