Exercise Sheet 3

1. Motivation of the geodesic equation

Let (M, g) be a compact Riemannian manifold and $c : [a, b] \to M$ a smooth curve parametrised by the arc length. Suppose that $c([a, b])$ is covered by one chart (ϕ, U). Construct m smooth variations $\gamma_\ell : (-\epsilon, \epsilon) \times [a, b] \to M$ such that the associated variation vector fields along c , $V_{\ell,0} := \big((\gamma_\ell)_* \frac{\partial}{\partial s}\big)(0, \cdot\,) \in \Gamma(c^*TM)$, satisfy that $\{V_{\ell,0}(t)\}_{1\leq\ell\leq m}$ is a basis of $TM_{c(t)}$ for all $t \in [a, b]$. Deduce that any smooth length-minimising curve is a geodesic (from the first variation of arc length [Theorem 1.15 in Prof. Lang's notes]).

2. Existence of closed geodesics

Let (M, g) be a compact Riemannian manifold and $c_0 \colon S^1 \to M$ a continuous closed curve. The purpose of this exercise is to show that in the family of all continuous and piece-wise C^1 curves $c\colon S^1 \to M$ which are homotopic to c_0, there is a shortest one and it is a geodesic.

a) Show that c_0 is homotopic to a piece-wise C^1-curve c_1 with finite length.

b) Let $L := \inf_c L(c)$ be the infimum over all piece-wise C^1 curves $c\colon S^1 \to M$ homotopic to c_0 and consider a minimizing sequence $(c_n \colon S^1 \to M)_n$ with $\lim_n L(c_n) = L$. Use compactness of M to construct a piece-wise C^1-curve $c\colon S^1 \to M$ with length L.

   Hint. Cover M with simply connected balls with the property that every two points in a ball are joined by a unique distance minimizing geodesic.

c) Conclude by showing that c is homotopic to c_0 and a geodesic.

3. Metric and Riemannian isometries

Let (M, g) and $(\bar{M}, \bar{g})$ be two connected Riemannian manifolds with induced distance functions d and $\bar{d}$, respectively. Further, let $f\colon (M, d) \to (\bar{M}, \bar{d})$ be an isometry of metric spaces, i.e. f is surjective and for all $p, p' \in M$ we have $\bar{d}(f(p), f(p')) = d(p, p')$.

a) Prove that for every geodesic γ in M, $\bar{\gamma} := f \circ \gamma$ is a geodesic in N.

b) Let $p \in M$. Define $F\colon TM_p \to T\bar{M}_{f(p)}$ with

$$F(X) := \left.\frac{d}{dt}\right|_{t=0} f \circ \gamma_X(t),$$

where γ_X is the geodesic with $\gamma_X(0) = p$ and $\dot{\gamma}(0) = X$. Show that F is surjective and satisfies $F(cX) = cF(X)$ for all $X \in TM_p$ and $c \in \mathbb{R}$.

c) Conclude that F is an isometry by proving $\|F(X)\| = \|X\|$.

d) Prove that F is linear and conclude that f is smooth in a neighborhood of p.

e) Prove that f is a diffeomorphism for which $f^*\bar{g} = g$ holds.