Exercise Sheet 4

1. Applications of Hopf-Rinow

(a) Let (M, g) be a *homogeneous Riemannian manifold*, i.e. the isometry group of M acts transitively on M. Prove that M is geodesically complete.

(b) Show that if (M, g) is a complete non-compact Riemannian manifold then there exist a ray emanating from any given $p \in M$, that is, a geodesic $c : [0, +\infty) \to M$ such that $c_v(0) = p$ and $\mathrm{dist}(p, c_v(t)) = t$ for all $t \geq 0$.

2. Ricci curvature

Let (M, g) be a 3-dimensional Riemannian manifold. Show the following:

a) The Ricci curvature ric uniquely determines the Riemannian curvature tensor R.

b) If M is an Einstein manifold, that is, a Riemannian manifold (M, g) with $\mathrm{ric} = \mathrm{kg}$ for some $k \in \mathbb{R}$, then the sectional curvature sec is constant.

3. Constant sectional curvature

Let (M, g) be a Riemannian manifold with constant sectional curvature $\sec(E) = \kappa \in \mathbb{R}$ for all $E \in G_2(TM)$. Show that

$$R(X, Y)W = \kappa \left(g(Y, W)X - g(X, W)Y \right).$$