Exercise Sheet 5

1. Divergence and Laplacian

Let (M, g) be a Riemannian manifold with Levi-Civita connection D. The *divergence* $\operatorname{div}(Y)$ of a vector field $Y \in \Gamma(TM)$ is the contraction of the $(1,1)$-tensor field $DY : X \mapsto D_X Y$ and the *Laplacian* $\Delta : C^\infty(M) \to C^\infty(M)$ is defined by $\Delta f := \operatorname{div}(\operatorname{grad} f)$. Show that:

a) $\operatorname{div}(fY) = Y(f) + f \operatorname{div} Y$,

b) $\Delta(fg) = f\Delta g + g\Delta f + 2\langle \operatorname{grad} f, \operatorname{grad} g\rangle$,

c) Compute Δf in local coordinates.

2. Codazzi equation

Let $M \subset \bar{M}$ be a submanifold of the Riemannian manifold $(\bar{M}, \bar{g})$. For the second fundamental form h of M, we define

$$(D^\perp_X h)(Y, W) := (\bar{D}_X(h(Y, W)))^\perp - h(D_X Y, W) - h(Y, D_X W),$$

where $W, X, Y \in \Gamma(TM)$. Show that the Codazzi equation

$$\big(\bar{R}(X, Y)W\big)^\perp = (D^\perp_X h)(Y, W) - (D^\perp_Y h)(X, W)$$

holds for all $W, X, Y \in \Gamma(TM)$.

3. Sectional curvature of submanifolds

Let $(\bar{M}, \bar{g})$ be a Riemannian manifold with sectional curvature $\bar{\sec}$. Let $p \in \bar{M}$ and $L \subset T\bar{M}_p$ an m-dimensional linear subspace.

a) Prove that there is some $r > 0$ such that for the open ball $B_r(0) \subset T\bar{M}_p$, the set $M := \exp_p(L \cap B_r(0))$ is an m-dimensional submanifold of $\bar{M}$.

b) Let g be the induced metric on M and let sec be the sectional curvature of M. Show that for $E \subset TM_p$, we have $\sec_p(E) = \bar{\sec}_p(E)$ and if L is a 2-dimensional subspace, then $\sec \le \bar{\sec}$ on M.