Exercise Sheet 6

1. Revisiting connections

(a) Fix some manifold M with a connection ∇. Take any $(1,2)$ tensor field F and define

$$\widetilde{\nabla}_X Y := \nabla_X Y + F(X,Y).$$

Show that

- $\widetilde{\nabla}$ is a connection
- For every connection $\hat{\nabla}$ on M there is a unique $(1,2)$ tensor $\hat{F}$ such that $\hat{\nabla} - \nabla = \hat{F}$. Show that in local coordinates $\hat{\Gamma}^k_{ij} - \Gamma^k_{ij} = F^k_{ij}$. Double check that the difference of two Christoffel symbols indeed transforms like a $(1,2)$ tensor field.

(b) Let $\nabla, \widetilde{\nabla}$ be two connections on M and $F(X,Y) := \widetilde{\nabla}_X Y - \nabla_X Y$ be their difference. Show that ∇ and $\widetilde{\nabla}$ have the same geodesics if and only if F is antisymmetric i.e., $F(X,Y) = -F(Y,X)$. Recall that a geodesic for ∇ is a self-parallel curve w.r.t. ∇, this translates in the ODE (with a harmless abuse of notation) $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$.

Conclude that if ∇ and $\widetilde{\nabla}$ have the same geodesics *and* the same torsion then $\nabla = \widetilde{\nabla}$.

2. Meaning of the torsion

Consider the manifold $(\mathbb{R}^3, g_{Eucl})$ endowed with the Levi-Civita connection ∇. Define another connection $\widetilde{\nabla}$ by

$$\widetilde{\nabla}_{\partial_i}\partial_j = \varepsilon^k_{ij}\partial_k \quad (\Longleftrightarrow \widetilde{\Gamma}^k_{ij} := \varepsilon^k_{ij}),$$

where ε^k_{ij} is the sign of the permutation $(1,2,3) \mapsto (i,j,k)$, and zero otherwise.

- Show that $\widetilde{\nabla}$ is a connection compatible with g, has the same geodesics of ∇, but has non-vanishing torsion. Check that

$$(\widetilde{\nabla}_v X)_p = (\nabla_v X)_p - X_p \times v.$$

- Compute the parallel transport of $v := (1,0,0)$ along $\gamma(t) := (0,0,t)$. This should clarify where is the "torsion".

- Show that, up to a multiplicative $C^\infty(\mathbb{R}^3)$ function in the Christoffels, $\widetilde{\nabla}$ is the unique connection that is g-compatible and has the same geodesics of ∇.

3. The sphere

Use Gauss' equations to prove that the sphere of radius $r > 0$,

$$\mathbb{S}^n_r := \{x \in \mathbb{R}^{n+1} \ : \ |x| = 1\}$$

has constant sectional curvatures equal to $1/r^2$.