Exercise Sheet 7

1. Jacobi fields in space forms

Let M be a space form with curvature $\kappa \in \mathbb{R}$. Furthermore, let $c\colon \mathbb{R} \to M$ be a geodesic which is parametrized by arc length and $N_0 \in TM_{c(0)}$ with $|N_0| = 1$, $\langle N_0, \dot{c}(0)\rangle = 0$. Determine the Jacobi field Y along c with starting conditions $Y(0) = aN_0$ and $\dot{Y}(0) = bN_0$ for $a, b \in \mathbb{R}$.

2. Trace of a symmetric bilinear form

Let $(V, \langle\cdot,\cdot\rangle)$ be a m-dimensional Euclidean space and let $r\colon V \times V \to \mathbb{R}$ be a symmetric bilinear form. Furthermore, let $S^{m-1} = \{v \in V : |v| = 1\}$ be the unit sphere. Prove that

$$\int_{S^{m-1}} r(v,v)\,\mathrm{dvol}^{S^{m-1}} = \frac{\mathrm{vol}(S^{m-1})}{m}\,\mathrm{tr}(r) = \omega_m\,\mathrm{tr}(r),$$

where $\mathrm{dvol}^{S^{m-1}}$ denotes the induced volume on S^{m-1} and ω_m is the volume of the m-dimensional unit ball.

3. Small balls and scalar curvature

Let p be a point in the m-dimensional Riemannian manifold (M, g). The goal is to prove the following Taylor expansion of the volume of the ball $B_r(p)$ as a function of r:

$$\mathrm{vol}(B_r(p)) = \omega_m r^m \left(1 - \frac{1}{6(m+2)}\,\mathrm{scal}(p) r^2 + \mathcal{O}(r^3)\right).$$

a) Let $v \in TM_p$ with $|v| = 1$, define the geodesic $c(t) \coloneqq \exp_p(tv)$ and let $e_1 = v, e_2, \dots, e_m \in TM_p$ be an orthonormal basis. Consider the Jacobi fields Y_i along c with $Y_i(0) = 0$ and $\dot{Y}_i(0) = e_i$ for $i = 2, \dots m$. Show that the volume distortion factor of $\exp_p$ at tv is given by

$$J(v,t) \coloneqq \sqrt{\det\left(\langle T_{tv}e_i, T_{tv}e_j\rangle\right)} = t^{-(m-1)}\sqrt{\det\left(\langle Y_i, Y_j\rangle\right)},$$

where $T_{tv} \coloneqq (d\exp_p)_{tv}$.

b) Let $E_2, \dots, E_m$ be parallel vector fields along c with $E_i(0) = e_i$. Then the Taylor expansion of Y_i is

$$Y_i(t) = tE_i - \sum_{k=2}^{m}\left(\tfrac{t^3}{6}R(e_i, v, e_k, v) + \mathcal{O}(t^4)\right)E_k.$$

c) Conclude that $J(v,t) = 1 - \frac{t^2}{6}\operatorname{ric}(v,v) + \mathcal{O}(t^4)$.
Hint: Use $\det(I_m + \epsilon A) = 1 + \epsilon \operatorname{tr}(A) + \mathcal{O}(\epsilon^2)$.

d) Prove the above formula for $\operatorname{vol}(B_r(p))$.