Exercise Sheet 8

1. Locally symmetric spaces

Let M be a connected m-dimensional Riemannian manifold. Then M is called *locally symmetric* if for all $p \in M$ there is a normal neighborhood $B(p, r)$ such that the *local geodesic reflection* $\sigma_p := \exp_p \circ (-\operatorname{id}) \circ \exp_p^{-1} \colon B(p, r) \to B(p, r)$ is an isometry.

(a) Show that if M is locally symmetric, then $DR \equiv 0$.

[Use that $d(\sigma_p)_p = -\operatorname{id}$ on TM_p.]

(b) Suppose that $DR \equiv 0$. Show that if $c \colon [-1, 1] \to M$ is a geodesic and $\{E_i\}_{i=1}^m$ is a parallel orthonormal frame along c, then $R(E_i, c')c' = \sum_{k=1}^m r_i^k E_k$ for *constants* r_i^k.

(c) Show that if $DR \equiv 0$, then M is locally symmetric.

[Let $q \in B(p, r), q \neq p$, and $v \in TM_q$. To show that $|d(\sigma_p)_q(v)| = |v|$, consider the geodesic $c \colon [-1, 1] \to B(p, r)$ with $c(0) = p, c(1) = q$, and a Jacobi field Y along c with $Y(0) = 0$ and $Y(1) = v$. Use (b).]

2. Conjugate points in manifolds with curvature bounded from above

(a) Prove directly, without using the Rauch Comparison Theorem, that there are no conjugate points in manifolds with non-positive sectional curvature.

(b) Show that in manifolds with sectional curvature at most κ, where $\kappa > 0$, there are no conjugate points along geodesics of length $< \pi/\sqrt{\kappa}$.

(c) Show that if $c \colon [0, \pi/\sqrt{\kappa}] \to M$ is a unit speed geodesic in a manifold with $\sec \geq \kappa > 0$, then some $c(t)$ is conjugate to $c(0)$ along $c|_{[0,t]}$.

3. Volume comparison

Let M be an m-dimensional Riemannian manifold with sectional curvature $\sec \leq \kappa$, $p \in M$ and $r > 0$ such that $\exp_p|_{B_r(0)}$ is a diffeomorphism. Furthermore, let $V_\kappa^m(r)$ denote the volume of a ball with radius r in the m-dimensional model space M_κ^m of constant sectional curvature $\kappa \in \mathbb{R}$. Prove that $V(B_r(p)) \geq V_\kappa^m$.