Exercise Sheet 9

1. Poincaré models of hyperbolic space

Let us introduce the following two well-known models of the hyperbolic space:

$$\text{Unit ball } \{|z|<1\} \subset \mathbb{R}^n \text{ equipped with metric } g_{ij} = \frac{4\delta_{ij}}{(1-|z|^2)^2}$$

and

$$\text{Half space } \{x^n > 0\} \subset \mathbb{R}^n \text{ equipped with metric } g_{ij} = \frac{\delta_{ij}}{(x^n)^2}.$$

a) Show that composing the transformations $y = x + (\frac{1}{2} - 2x^n)\boldsymbol{e}_n$ and $z = \boldsymbol{e}_n + (y - \boldsymbol{e}_n)|y - \boldsymbol{e}_n|^{-2}$ give an isometry between the two previous Riemannian manifolds

b) Show that, for the second model, circular arcs at $\{x^n = 0\}$ are geodesics.

c) Show that given any given point all geodesic rays $x(t)$, $t \geq 0$ emanating from it are minimizing up to arbitrarily large values of $t > 0$ (note that this is stronger than geodesic completeness).

d) Show that the sectional curvatures are constantly equal to -1.

2. "Uniqueness" and symmetries of the hyperbolic space

Prove that if M is a n-dimensional Riemannian manifold satisfying properties c) and d) in the previous exercise and $p \in M$ then $\exp_p$ induces an isometry between $\mathbb{R}^n$ with metric

$$g(w,w) = \left(w \cdot \tfrac{x}{|x|}\right)^2 + \left(|w|^2 - \left(w \cdot \tfrac{x}{|x|}\right)^2\right)\frac{\sinh^2|x|}{|x|^2} \tag{1}$$

and M. Deduce that given any two points p, q in the hyperbolic space $\mathbb{H}$ and any isometry between their tangent spaces $T\mathbb{H}_p \to T\mathbb{H}_q$ there is a unique isometry $f : \mathbb{H} \to \mathbb{H}$ such that $f(p) = q$ and $df_p = H$.

3. Translations

Suppose that Γ is a group of translations of $\mathbb{R}^m$ that acts freely and properly discontinuously on $\mathbb{R}^m$.

a) Show that there exist linearly independent vectors $v_1, \dots, v_k \in \mathbb{R}^m$ such that

$$\Gamma = \left\{ x \mapsto x + \sum_{i=1}^{k} z_i v_i : (z_1, \dots, z_k) \in \mathbb{Z}^k \right\} \simeq \mathbb{Z}^k.$$

b) Let l denote the infimum of the lengths of all closed curves in $\mathbb{R}^m/\Gamma$ that are not null-homotopic. Show that l equals the length of the shortest non-zero vector of the form $\sum_{i=1}^{k} z_i v_i$ with $z_i \in \mathbb{Z}$ as above.