Solutions 1

1. Divergence Theorem

Let $M \subset \mathbb{R}^3$ be a compact 3-dimensional manifold with boundary, $N\colon \partial M \to S^2$ the outward pointing unit normal,

$$\pi = f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$$

a 2-form on $\mathbb{R}^3$ and $X = (f, g, h)$.

(a) Show that $d\pi = \operatorname{div}(X) dx \wedge dy \wedge dz$.

(b) Deduce the Divergence Theorem

$$\int_M \operatorname{div}(X)\, \mathrm{dVol} = \int_{\partial M} \langle X, N\rangle\, \mathrm{d}A$$

from the Theorem of Stokes for differential forms.

Solution.
(a)

$$\begin{aligned} d\pi &= \frac{\partial f}{\partial x} dx \wedge dy \wedge dz + \frac{\partial g}{\partial y} dy \wedge dz \wedge dx + \frac{\partial h}{\partial z} dz \wedge dx \wedge dy \\ &= \Big(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\Big) dx \wedge dy \wedge dz \\ &= \operatorname{div}(X) dx \wedge dy \wedge dz. \end{aligned}$$

(b) Let (U, φ) be a chart of ∂M such that for (the parametrization) $\psi := \varphi^{-1}\colon \varphi(U) \subset \mathbb{R}^2 \to \partial M$ the outward pointing unit normal is given by

$$N \circ \psi = \frac{\partial_x \psi \times \partial_y \psi}{\|\partial_x \psi \times \partial_y \psi\|} = \frac{\psi_1 \times \psi_2}{\|\psi_1 \times \psi_2\|} = \frac{d\psi(e_1) \times d\psi(e_2)}{\|d\psi(e_1) \times d\psi(e_2)\|}.$$

Then

$$\int_U \pi = \int_{\varphi(U)} \psi^* \pi.$$

In order to integrate $\psi^*\pi$ we need to write it in the form $\psi^*\pi = a dx \wedge dy$ and find the coefficient function $a\colon \varphi(U) \to \mathbb{R}$.

Recall that for $q = \varphi(p)$, $a(q) = (a dx \wedge dy)_q(e_1, e_2)$ thus

$$
\begin{aligned}
a(q) &= (\psi^*\pi)_q(e_1, e_2) \\
&= \pi_p(d\psi_q(e_1), d\psi_q(e_2)) \\
&= \pi_p(\psi_1(q), \psi_2(q)) \\
&= f(p) \underbrace{\det \begin{pmatrix} \psi_1^2 & \psi_2^2 \\ \psi_1^3 & \psi_2^3 \end{pmatrix} (q)}_{=\left(\psi_1(q)\times\psi_2(q)\right)^1} + g(p) \underbrace{\det \begin{pmatrix} \psi_1^3 & \psi_2^3 \\ \psi_1^1 & \psi_2^1 \end{pmatrix} (q)}_{\left(\psi_1(q)\times\psi_2(q)\right)^2} + h(p) \underbrace{\det \begin{pmatrix} \psi_1^1 & \psi_2^1 \\ \psi_1^2 & \psi_2^2 \end{pmatrix} (q)}_{\left(\psi_1(q)\times\psi_2(q)\right)^3} \\
&= \langle X(p), \psi_1(q) \times \psi_2(q) \rangle \\
&= \langle X \circ \psi(q), \frac{\psi_1(q) \times \psi_2(q)}{\|\psi_1(q) \times \psi_2(q)\|} \rangle \cdot \|\psi_1(q) \times \psi_2(q)\| \\
&= \langle X, N \rangle \circ \psi(q) \cdot \|\psi_1(q) \times \psi_2(q)\|.
\end{aligned}
$$

It's a computation to see that $\|\psi_1(q) \times \psi_2(q)\| = \sqrt{\det \langle \psi_i(q), \psi_j(q) \rangle_{i,j=1,2}}$, so

$$
\int_U \pi = \int_{\varphi(U)} \psi^*\omega = \int_{\varphi(U)} \langle X, N \rangle \circ \psi \cdot \|\psi_1(q) \times \psi_2(q)\| \, \mathrm{d}x \, \mathrm{d}y = \int_U \langle X, N \rangle \, \mathrm{d}A.
$$

Using (a) and the theorem of Stokes (together with a partition of unity) we obtain

$$
\int_M \mathrm{div} X \, \mathrm{dVol} = \int_M \mathrm{div}(X) dx \wedge dy \wedge dz = \int_M d\pi = \int_{\partial M} \pi = \int_{\partial M} \langle X, N \rangle \, \mathrm{d}A.
$$

2. Orthogonal Structures

Let $\pi\colon E \to M$ be a vector bundle of rank k over a manifold M. An *orthogonal structure* g on E assigns to every point $p \in M$ a scalar product g_p on the fiber $E_p := \pi^{-1}(p)$, such that for all sections s, s' the map $p \mapsto g_p(s(p), s'(p))$ is smooth.

Prove that every vector bundle admits an orthogonal structure.

Hint: Use a partition of unity.

Solution. We fix a bundle atlas $\{\psi_\alpha\colon V_\alpha := \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k\}_{\alpha\in A}$ with $\psi_\alpha = (\pi, h_\alpha)$. Define an orthogonal structure

$$g_p^\alpha(\xi,\eta) := \langle h_\alpha(\xi), h_\alpha(\eta)\rangle,$$

on V_α where $\langle\cdot,\cdot\rangle$ denotes the standard scalar product on $\mathbb{R}^k$.

Consider now a partition of unity $\{\lambda_\alpha\colon M \to \mathbb{R}\}_{\alpha\in A}$ subordinate to $\{U_\alpha\}_{\alpha\in A}$. Then

$$g_p(\xi,\eta) := \sum_{\alpha\in A} \lambda_\alpha(p)\cdot g_p^\alpha(\xi,\eta)$$

defines an orthogonal structure on E. Indeed, $\lambda_\alpha \geq 0$, for every $p \in M$ there exists $\alpha \in A$ with $\lambda_\alpha(p) > 0$ and the sum is locally finite; therefore g_p is a scalar product.

Moreover

$$p \mapsto g_p(s(p), s'(p)) = \sum_{\alpha\in A} \lambda_\alpha(p)\cdot\langle h_\alpha\circ s(p), h_\alpha\circ s'(p)\rangle$$

is smooth as composition of smooth maps.

3. Vector Bundles of Rank 1

a) Prove that every vector bundle of rank 1 over a simply connected manifold is trivial.

b) Prove that, up to isomorphism, there exist exactly two vector bundles of rank 1 over S^1.

Solution. Let E be a vector bundle of rank 1 over M. From Exercise 2 we can choose a metric g on E and consider the subset $S := \{v \in E : g(v,v) = 1\}$.

Since E is a vector bundle of rank 1, the metric g is locally given by

$$g\colon U \times \mathbb{R}^2 \to \mathbb{R},\ g_p(v,w) = \theta(p)vw,$$

for an open subset $U \subset M$ and $\theta \in C^\infty(U)$ with $\theta(p) > 0$.[1]

Then S is given over U by the graph (not the trace!) of the smooth maps $p \mapsto v = \pm 1/\sqrt{\theta(p)}$. From this it follows that $\pi|_S\colon S \to M$ is a 2-covering of M.

a) We fix a point $p \in M$ and a vector $v \in S \cap \pi^{-1}(p)$. For every point $q \in M$ we choose a curve $\gamma_q\colon [0,1] \to M$ connecting p to q. Since $\pi|_S\colon S \to M$ is a covering, there is a unique lift $\bar\gamma_q\colon [0,1] \to S$ with $\bar\gamma_q(0) = v$. We define a map $s\colon M \to S$ of the vector bundle by setting $s(q) := \bar\gamma_q(1)$.

The map s is well defined because M is simply connected and hence curves starting and ending at the same points are homotopic. Indeed, let $\gamma'_q\colon [0,1] \to M$ be another curve as above. Since M is simply connected, there exists an homotopy $H\colon [0,1] \times [0,1] \to M$ with $H(0,t) = \gamma_q(t)$, $H(1,t) = \gamma'_q(t)$, $H(s,0) = p$ and $H(s,1) = q$. By the Homotopy Lifting Property there exists a unique lift $\bar H\colon [0,1] \times [0,1] \to S$ with $\bar H(0,t) = \bar\gamma_q(t)$ and with fixed end points. Then

$$\bar\gamma'_q(1) = \bar H(1,1) = \bar H(0,1) = \bar\gamma_q(1)$$

and s is well defined.

Moreove $s\colon M \to E$ is a smooth section, since locally S is given by the graph of a smooth function, as we have seen above.

Since s never vanishes, it follows from Proposition 10.3 (of the DGI notes) that the vector bundle is trivial.

[1] This is an expression for g on the image of the local trivializations of the vector bundle. Since the discussion is local and local trivializations are diffeomorphisms we can work there from now on. Recall that local trivializations restrict to vector space isomorphisms on the fibers of the bundle and therefore for each p, the image of g written above must be a multiple of the standard scalar product.

b) Let $\pi\colon E \to S^1$ be a vector bundle of rank 1 over S^1 and denote by S the 2-covering as above. Choose $p \in S^1$, $v \in S \cap \pi^{-1}(p) = \{v, -v\}$ and a simply closed curve $\gamma\colon [0,1] \to S^1$ with $\gamma(0) = \gamma(1) = p$. Consider now the unique lift $\bar{\gamma}\colon [0,1] \to S$ with $\bar{\gamma}(0) = v$. By definition of a lift there are now two possibilities, either $\bar{\gamma}(1) = v$ or $\bar{\gamma}(1) = -v$.

In the first case $\bar{\gamma}$ induces a nowehere vanishing section of the vector bundle, which maps every point $q \in S^1$ to the unique $w \in S \cap \pi^{-1}(q)$ lying on the trace of $\bar{\gamma}$. E is then trivial by Proposition 10.3.

In the second case, suppose that E is trivial. We want to reach a contradiction. Then by Proposition 10.3 there exists a non-trivial section $s\colon S^1 \to E$ and we can assume that $g_p(s(p), v) > 0$ (otherwise take the "opposite" section). Define $\widetilde{\gamma}\colon [0,1] \to S$ by

$$\widetilde{\gamma}(t) := \frac{s \circ \gamma(t)}{|s \circ \gamma(t)|_g} \in S \cap \pi^{-1}(\gamma(t))$$

and notice that $\widetilde{\gamma}$ is a lift of γ with $\widetilde{\gamma}(0) = v$ (we can rescale the value of $s(q) \in E_q$ along the fiber, e.g. dividing by its g_q-norm, without changing its projection onto S^1.). But $\widetilde{\gamma}(1) = v \neq -v = \bar{\gamma}(1)$. This contradicts the uniqueness of $\bar{\gamma}$ and therefore E is not trivial. Overall this shows that E is trivial if and only if $\bar{\gamma}(1) = v$.

Now we are finally ready to prove *b*). Either the vector bundle is trivial, and in that case we are done, or it's not.

A possible not trivial vector bundle of rank 1 over S^1 is given by

$$E := \left\{ \left(\cos t, \sin t, r\cos\tfrac{t}{2}, r\sin\tfrac{t}{2}\right) \in S^1 \times \mathbb{R}^2 : t, r \in \mathbb{R} \right\}$$

and $\pi\colon E \to S^1$, $\pi(x, y, u, v) := (x, y)$. It's not trivial because any lift of γ starting in (p, v) ends in $(p, -v)$.

It remains to show that any two not trivial rank 1-vector bundles $\pi\colon E \to S^1$ and $\pi'\colon E' \to S^1$ are isomorphic. Let $\bar{\gamma}\colon [0,1] \to S$ and $\bar{\gamma}'\colon [0,1] \to S'$ the two lifts of $\gamma\colon [0,1] \to S^1$ with $\bar{\gamma}(0) = v$ and $\bar{\gamma}(1) = -v$, respectively., $\bar{\gamma}'(0) = v'$ and $\bar{\gamma}'(1) = -v'$

Then $\Phi\colon E \to E'$, $\Phi(r\bar{\gamma}(t)) := r\bar{\gamma}'(t)$ is an isomorphism.