Solutions 10

1. Two dimensional Hadamard manifolds

Let (M, g) be a two dimensional Hadamard manifold. For fixed point $p \in M$ and isometry $H : \mathbb{R}^2 \to TM_p$, consider $(\mathbb{R}^2, \overline{g})$ where $\overline{g} := (\exp_p \circ H)^* g$.

(a) Show that $\overline{g}$ is of the form

$$\overline{g}_x(v, w) := \big(v \cdot \tfrac{x}{|x|}\big)\big(w \cdot \tfrac{x}{|x|}\big) + \frac{f^2(x)}{|x|^2}\Big(v \cdot w - \big(v \cdot \tfrac{x}{|x|}\big)\big(w \cdot \tfrac{x}{|x|}\big)\Big), \quad (1)$$

where $f^2(x)/|x|^2$ is smooth (also at $x = 0$) and has limit 1 as $x \to 0$, and where $t \mapsto f(tx)$ is nonnegative and convex for any fixed $x \in \mathbb{R}^2 \setminus \{0\}$.

(b) Reciprocally, show that $\mathbb{R}^2$ endowed with any metric $\overline{g}$ satisfying the properties established in (a) —and such that $g_x(v, w)$ extends to a smooth metric across $x = 0$— gives a model of a Hadamard manifold (simply connected with nonpositive sectional curvature at all points).

Solution:. (a) Fix $x \in \mathbb{R}^2 \setminus 0$ and let $c_x(t)$ be a geodesic emanating from p, with unit initial velocity $H(x)/|x| \in TM_p$. Let $E(t)$ be a parallel unit vector field along c_x which is orthogonal to $c_x'(t)$. Notice that $Y := \phi(t)E$ is a Jacobi field if, and only if, $\phi'' + (K \circ c_x)\phi = 0$. Now, for fixed $x \neq 0$ and let $w \in \mathbb{R}^2$ be perpendicular to x, we have

$$Y(t) := d(\exp_p \circ H)_{tx/|x|}(tw)$$

is a Jacobi field satisfying $Y(0) = 0$ and $Y'(0) = w$. Hence, by definition pullback metric $\overline{g} := (\exp_p \circ H)^* g$, for $t > 0$ we have

$$\begin{aligned}\overline{g}_{tx/|x|}(w, w) &= g\big(d(\exp_p \circ H)_{tx/|x|}(w), d(\exp_p \circ H)_{tx/|x|}(w)\big)\\ &= t^{-2} g\big(Y(t), Y(t)\big) = t^{-2} g\big(|w|\phi E(t), |w|\phi E(t)\big) = (\phi/t)^2 |w|^2,\end{aligned}$$

where ϕ is the unique solution of $\phi'' + (K \circ c_x)\phi = 0$ with initial conditition $\phi(0) = 0$ and $\phi'(0) = 1$.

Hence, setting $t = |x|$ in the equation above and defining $f(x) = \phi(|x|)$ as the unique solution $\phi'' + (K \circ c_x)\phi = 0$ with $\phi(0) = 0$ and $\phi'(0) = 1$ evaluated at time $t = |x|$, we obtain

$$\overline{g}_x(w, w) = (f(x)/|x|)^2 |w|^2,$$

Using that from Gauss' lemma

$$\overline{g}_x(v, w) = v \cdot w$$

whenever v parallel to x (and w is any vector), we obtain (1).

Finally observer that $\phi'' = -(K \circ c_x)\phi \geq 0$ implies that ϕ is convex (and hence so is $t \mapsto f(tx)$). Also, by l'Hopital's rule, $\lim_{t\to 0} \phi(t)/t = \phi'(0) = 1$ and hence the limit of $f^2(x)/|x|^2 \to 1$ as $x \to 0$.

(b) Consider now $\mathbb{R}^2$ endowed with a metric of the form (1). Take polar corrdinates (r, θ) in $\mathbb{R}^2 \setminus 0$. Notice that coordinates the metric is of the form

$$(g_{i,j}) = \begin{pmatrix} 1 & 0 \\ 0 & E \end{pmatrix}$$

where $E = E(r, \theta) = \overline{g}(\partial_\theta, \partial_\theta) = f^2(r\cos\theta, r\sin\theta)$.

The condition that f is convex along rays from 0 reads $\left(\sqrt{E}\right)_{11} \geq 0$.

In order to compute the curvature, let us compute the Chistoffel symbols (we still use polar coordinates). The only nonzero ones are:

$$\Gamma^2_{22} = \frac{E_2}{2E}, \quad \Gamma^2_{12} = \Gamma^2_{12} = \frac{E_1}{2E}, \quad \Gamma^1_{22} = \frac{-E_1}{2}$$

Hence, direct computation shows:

$$K = \frac{E_1^2}{4E^2} - \frac{E_{11}}{2E} = -\frac{\left(\sqrt{E}\right)_{11}}{\sqrt{E}} \leq 0.$$

2. Some consequences of non-positive sectional curvature

Let M be a Hadamard manifold. Prove the following:

(a) For each $p \in M$, the map $(\exp_p)^{-1}\colon M \to TM_p$ is 1-Lipschitz.

(b) For $p, x, y \in M$, it holds

$$d(p,x)^2 + d(p,y)^2 - 2d(p,x)d(p,y)\cos\gamma \leq d(x,y)^2,$$

where γ denotes the angle in p.

(c) Let m denote the midpoint of the geodesic xy in M and let $p \in M$. Then we have

$$d(p,m)^2 \leq \frac{d(p,x)^2 + d(p,y)^2}{2} - \frac{1}{4}d(x,y)^2.$$

Hint: Prove it first in the Euclidean plane. (This is a rather difficult but interesting to do exercise if one uses only the results up to Chapter 4 in Prof. Lang's notes. The exercise becomes simpler if one uses the content of Chapter 5.)

Solution: (a) By the Theorem of Hadamard-Cartan, we know that $\exp_p$ is a diffeomorphism, hence $(\exp_p)^{-1}$ is well defined.

For $x, y \in M$, let $\overline{x} := (\exp_p)^{-1}(x)$ and $\overline{y} := (\exp_p)^{-1}(y)$. Furthermore, let $c\colon [0,l] \to M$ be the (minimizing) geodesic from x to y. Then by Corollary 3.19, we get

$$d(\overline{x}, \overline{y}) \leq L((\exp_p)^{-1} \circ c) \leq L(c) = d(x,y).$$

(b) We denote again $\overline{x} := (\exp_p)^{-1}(x)$ for $x \in M$. Then this follows directly from Exercise 2(a) and the law of cosines in TM_p:

$$\begin{aligned} d(x,y)^2 &\geq d(\overline{x}, \overline{y})^2 \\ &= d(\overline{p}, \overline{x})^2 + d(\overline{p}, \overline{y})^2 - 2d(\overline{p}, \overline{x})d(\overline{p}, \overline{y})\cos\overline{\gamma} \\ &= d(p,x)^2 + d(p,y)^2 - 2d(p,x)d(p,y)\cos\gamma. \end{aligned}$$

A direct solution of Exercise 2(c). Given a geodesic triangle $xyz \in M$, a comparison triangle in $\mathbb{R}^2$ is a triangle $\overline{xyz}$ with sides of the same length. Give $q \in xy$, its comparison point in $\overline{xyz}$ is $\overline{q} \in \overline{xy}$ with $|\overline{xq}| = |xq|$. We will show that for all $p \in xy$ and $q \in xz$, $d(p, q) \leq d(\overline{p}, \overline{q})$. [1]

Step 1: Let $p \in M$, $v, w \in TM_p$ and consider the geodesic triangle with vertices $x := \exp_p(v)$, $y := \exp_p(w)$ and two sides $\exp_p(tv)$, $\exp_p(tw)$ Let $\overline{p}, \overline{x}, \overline{y}$ be a comparison triangle in $\mathbb{R}^2$.

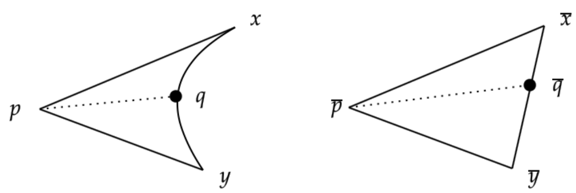
Then we know that $d(v, w) \leq d(x, y)$ and $\measuredangle_p(x, y) = \measuredangle_0(v, w)$, so $\measuredangle_{\overline{p}}(\overline{x}, \overline{y}) \geq \measuredangle_0(v, w) = \measuredangle_p(x, y)$

In general for a geodesic triangle xyz in M with internal angles α, β, γ and a comparison triangle $\overline{x}, \overline{y}, \overline{z}$ with internal angles $\overline{\alpha}, \overline{\beta}, \overline{\gamma}$ it holds that

$$\alpha \leq \overline{\alpha}, \quad \beta \leq \overline{\beta}, \quad \gamma \leq \overline{\gamma}.$$

Step 2: Let p, x, y be a geodesic triangle in M and $\overline{p}, \overline{x}, \overline{y}$ be a comparison triangle in $\mathbb{R}^2$. Let $q \in xy$ be any point and $\overline{q} \in \overline{xy}$ its comparison point in the comparison triangle ($d(x, q) = d(\overline{x}, \overline{q})$), then

$$d(p, q) \leq d(\overline{p}, \overline{q}).$$

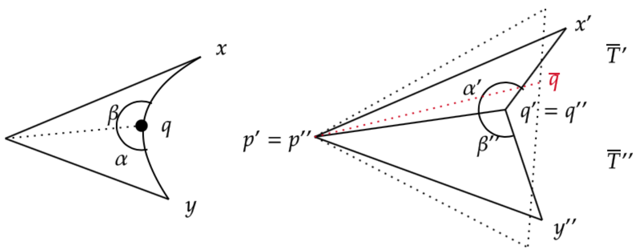


Consider the geodesic triangles $T' = pqx$ and $T'' = pqy$ and their respective comparison triangles $\overline{T}' = p'q'x'$ and $\overline{T}'' = p''q''y''$ in $\mathbb{R}^2$. Since $d(p, q) = d(p', q') = d(p'', q'')$, we can assume that $p' = p''$, $q' = q''$ and $\overline{T}', \overline{T}''$ lie on the opposide side of the straight segment $p'q'$. Denote

$$\alpha := \measuredangle_q(p, x) \qquad \beta := \measuredangle_q(p, y)$$
$$\alpha' := \measuredangle_{q'}(p', x') \qquad \beta'' := \measuredangle_{q''}(p'', y'').$$

By the previous step $\alpha' + \beta'' \geq \alpha + \beta = \pi$. This implies that in order to make $\overline{T}' \cup \overline{T}''$ a comparison triangle for pxy we need to increase the distance from $\overline{p}$ to $\overline{q}$, hence $d(\overline{p}, \overline{q}) \geq d(p, q)$.

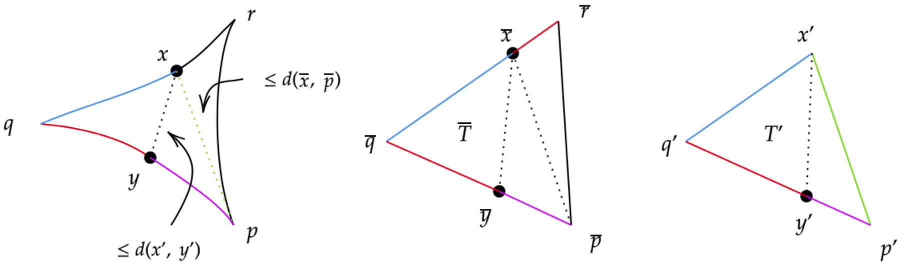
[1] In other words Hadamard manifolds are CAT(0)-spaces.



Step 3: Let M be a Hadamard manifold and let p, q, r be a geodesic triangle in M. Let $x \in qr$, $y \in qp$ be any two points on its sides, then we claim that

$$d(x, y) \leq d(\overline{x}, \overline{y})$$

where $\overline{x}, \overline{y}$ are comparison points on the sides of a comparison triangle $\overline{p}, \overline{q}, \overline{r}$ for pqr in $\mathbb{R}^2$.



Consider the geodesic triangle qxp and its comparison triangle T' with vertices $q'x'p'$. Denote by y' the comparison point of y on T'. Now, on one hand we have $d(q', p') = d(\overline{p}, \overline{q})$, $d(q', x') = d(\overline{q}, \overline{p})$ while on the other hand, the previous step implies $d(x', p') \leq d(\overline{x}, \overline{p})$ and therefore $d(x', y') \leq d(\overline{x}, \overline{y})$. As $d(x, y) \leq d(x', y')$ by the previous step, we obtain the claim.

Step 4: We can rewrite the conclusion of the previous lemma in the following way. Let M be a Hadamard manifold. Let $\sigma, \sigma' \colon [0, 1] \to M$ be two geodesic with $\sigma(0) = \sigma'(0)$ and consider the triangle $x := \sigma(0), y := \sigma(1), z := \sigma'(1)$ in M. Let $0, v, v'$ be a comparison triangle in $\mathbb{R}^2$ with $|v| = d(\sigma(0), \sigma(1))$, $|v'| = d(\sigma'(0), \sigma'(1))$ and $|v - v'| = d(\sigma(1), \sigma'(1))$.

By the previous step

$$d\big(\sigma(t), \sigma'(t')\big) \leq |tv - t'v'|,$$

in particular $d\big(\sigma(t), z\big) \leq |tv - v'|$. Furthermore $|tv - v'|^2 - t|v - v'|^2 = t(t-1)|v|^2 + (1-t)|v'|^2$, so if we denote $m = \sigma(\frac{1}{2})$, the midpoint of xy we obtain

$$d(m, z) \leq \frac{1}{2}\Big(d(x, z)^2 + d(y, z)^2\Big) - \frac{1}{4} d(x, y)^2.$$

3. Isometries with bounded orbits.

Let M be a Hadamard manifold. Prove the following:

(a) If $Y \subset M$ is a bounded set, then there is a unique point $c_Y \in M$ such that $Y \subset \overline{B}(c_Y, r)$, where $r := \inf\{s > 0 : \exists x \in M \text{ such that } Y \subset \overline{B}(x, s)\}$.

Hint: Prove it first in Euclidean space. It may be useful to use part (c) of exercise 2. We call c_Y the *center* of Y.

(b) Let γ be an isometry of M. Then γ is elliptic if and only if M has a bounded orbit. Furthermore, if γ^n is elliptic for some integer $n \neq 0$, then γ is elliptic.

Solution: (a) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points such that $Y \subset \overline{B}(x_n, r_n)$ for $r_n > 0$ with $r_n \to r$. We claim that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Then it follows that the sequence converges to some $c_Y \in M$ with the required property and the fact that every such sequence is Cauchy establishes uniqueness.

Fix some $\epsilon > 0$ and let $N \in \mathbb{N}$ such that $r_n < r + \epsilon$ for all $n \geq N$. For $n, n' \geq N$ let m be the midpoint of x_n and $x_{n'}$. Then by the definition of r, there is some $y \in Y$ such that $d(m, y) \geq r$. By Exercise 2(c), we therefore get

$$\begin{aligned}\frac{1}{4}d(x_n, x_{n'})^2 &\leq \frac{d(y, x_n)^2 + d(y, x_{n'})^2}{2} - d(y, m)^2 \\ &\leq \frac{r_n^2 + r_{n'}^2}{2} - r^2 \\ &\leq (r + \epsilon)^2 - r^2 = 2r\epsilon + \epsilon^2.\end{aligned}$$

(b) If $Y = \{\gamma^k x : k \in \mathbb{Z}\}$ is a bounded orbit of γ, then Y is γ invariant and hence c_Y is a fixed point.

Let $x \in M$ be a fixed point of γ^n. Then $Y = \{\gamma^k x : k \in \mathbb{Z}\}$ is finite and therefore bounded.