Solutions 11

1. Asymptotic expansion of the circumference

Let M be a manifold, $E \subset TM_p$ a linear 2-plane and $\gamma_r \subset E$ a circle with center 0 and radius $r > 0$ sufficiently small. Show that

$$L(\exp(\gamma_r)) = 2\pi \left(r - \frac{\sec(E)}{6} r^3 + \mathcal{O}(r^4) \right)$$

for $r \to 0$.

Solutions. Let $v, w \in TM_p$ be an orthonormal basis of E. Then the circle can be parametrized by $\gamma_r(\varphi) = r(v \cos\varphi + w \sin\varphi)$. For some fixed $\varphi_0 \in [0, 2\pi]$, consider the Jacobi field $Y_{\varphi_0}(r)$ associated to the geodesic variation $V(\varphi, r) := \exp(\gamma_r(\varphi))$ of the geodesic $c_{\varphi_0}(r) := \exp(\gamma_r(\varphi_0))$. Then it holds

$$L(\exp(\gamma_r)) = \int_0^{2\pi} |Y_\varphi(r)| \, d\varphi.$$

We will now compute the Taylor expansion for $|Y_0(r)|$ (compare with Serie 7, Exercise 3), all other cases are similar. We have $Y_0(0) = 0$ and $Y_0'(0) = w$. From the Jacobi equation we also get

$$Y_0''(0) = -R\left(Y_0, c_0'\right) c_0' \Big|_{r=0} = 0.$$

Now taking the derivative of the Jacobi equation, we get

$$Y_0'''(0) = -\frac{D}{dr} R\left(Y_0, c_0'\right) c_0' \Big|_{r=0} = -R\left(Y_0', c_0'\right) c_0' \Big|_{r=0} = -R(w, v)v.$$

It follows that

$$|Y_0(r)| = r - \frac{R(w, v, w, v)}{6} r^3 + \mathcal{O}(r^4).$$

Therefore, we finally get

$$L(\exp(\gamma_r)) = \int_0^{2\pi} \left(r - \frac{\sec(E)}{6} r^3 + \mathcal{O}(r^4) \right) d\varphi = 2\pi \left(r - \frac{\sec(E)}{6} r^3 + \mathcal{O}(r^4) \right),$$

as it was to show.

2. Isoperimetric problem in two dimensional Hadamard manifolds

Let M be a 2-dimensional Hadamard manifold. Given $\Omega \subset M$ bounded, we say that $\partial\Omega$ is C^2 if it consists of a finite disjoint union of C^2 simple close curves. For such Ω define the *isoperimetric quotient*

$$\mathcal{I}(\Omega) := \frac{\text{length}(\partial\Omega)}{\text{area}(\Omega)^{\frac{1}{2}}}.$$

a) Suppose first that M is isometric to the Euclidean plane. Show that if Ω_0 is a minimizer of $\mathcal{I}$ (such that $\partial\Omega_0$ is C^2) then

$$\mathcal{I}(\Omega_0) = \sqrt{4\pi} \quad \text{and } \Omega_0 \text{ is an Euclidean disc.}$$

Hint: Show that a smooth minimizer $\partial\Omega_0$ must consist of exactly simple curve γ, and prove (using the first variation of arc length) that the geodesic curvature κ_g of γ must be constant. Deduce that γ must trace a circle in $\mathbb{R}^2$.

b) In the case of nonnegative Gauss curvature $K \leq 0$, show that if Ω_0 is a minimizer of $\mathcal{I}$ (with $\partial\Omega_0$ of class C^2) then $\mathcal{I}(\Omega_0) = \sqrt{4\pi}$, and Ω_0 is isometric to an Euclidean ball.

Hint: Using small metric balls $B_r(p) \subset M$, with $r \ll 1$ as "competitors", prove that $\mathcal{I}(\Omega_0) \leq \sqrt{4\pi}$. Show that, as in a), $\partial\Omega_0$ must consist of only one closed simple curve γ. Let ν be the inwards unit normal to $\partial\Omega_0$, define (for ε small) $\gamma_\varepsilon(t) := \gamma(t) + \varepsilon\nu(t)$, and let Ω_ε be the bounded connected component of $M \setminus \text{image}(\gamma_\varepsilon)$. Show that $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} I(\Omega_\varepsilon) \leq 0$, and < 0 unless $K \equiv 0$ in Ω_0.

Solutions. a) Let $M = \mathbb{R}^2$ with Euclidean metric. Note that if Ω_0 has multiple components each is a closed simple curve. Hence, the image of each of these curves it divides $\mathbb{R}^2$ into two connected components (one bounded and one unbounded). Now, the union of (the closures of) the bounded components is a new set which contains Ω_0 and whose boundary is contained in $\partial\Omega_0$. Hence, this set obtained by "filling the holes" it would have more area and less perimeter, contradicting the fact that Ω_0 minimizes $\mathcal{I}$.

Let $\gamma : (0, L) \to \mathbb{R}^2$ be a curve tracing $\partial\Omega_0$, parametrized by the arc length, and let $\nu : [0, L] \to \mathbb{S}^1$ be the inwards unit normal. Given $\xi \in C^2_{\text{closed}}([a, b])$ define $\gamma_\varepsilon(t) := \gamma(t) + \varepsilon\xi(t)\nu(t)$ and let Ω_ε be the bounded connected component of $\mathbb{R}^2 \setminus \text{image}(\gamma_\varepsilon)$. If $\int_0^L \xi(t) = 0$ then $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \text{area}(\Omega_\varepsilon) = 0$. Hence be minimality it must be $\frac{d}{d\varepsilon}\big|_{\varepsilon=0} \text{length}(\Omega_\varepsilon) = \int_0^L \kappa_g(t)\xi(t)dt = 0$. Since ξ is an arbitrary average zero smooth function we deduce that $\kappa_g \equiv \kappa = \textit{constant}$ or equivalently $c'' \equiv \kappa\nu$. This easily implies that c traces a circle with radius $1/\kappa$.

b) Using e.g. Exercice 1, we obtain that, for all $p \in M$, $\text{area}(B_r(p)) = \pi r^2(1+\mathcal{O}(r^2))$ and $\text{length}(\partial B_r(p)) = 2\pi r(1+\mathcal{O}(r^2))$ as $r \downarrow 0$. This gives $\mathcal{I}(\Omega_0) \le \sqrt{4\pi}$.

As in a) —using that M diffeomorphic to $\mathbb{R}^2$—, $\partial\Omega_0$ must consist of only one simple closed curve γ. Let ν be the inwards unit normal to $\partial\Omega_0$ let $\kappa_g := \langle c'', \nu\rangle$ be the geodesic curvature, where $c'' = \frac{D}{dt}c'$. Define (for ε small) $\gamma_\varepsilon(t) := \gamma(t) + \varepsilon\nu(t)$, and let Ω_ε be the bounded connected component of $M \setminus \text{image}(\gamma_\varepsilon)$. Let us show that show that $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\mathcal{I}(\Omega_\varepsilon) \le 0$, and < 0 unless $K \equiv 0$ in Ω_0.

Indeed, on the one hand $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\text{area}(\Omega_\varepsilon) = -\text{length}(\partial\Omega_0)$. On the other hand, $\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\text{length}(\Omega_\varepsilon) = -\int_{\partial\Omega_0}\kappa_g ds$

Now, using Gauss-Bonnet, $\int_{\partial\Omega_0}\kappa_g ds = 2\pi - \int_{\Omega_0} K dA \ge 2\pi$ ($>2\pi$ unless $K \equiv 0$). Hence,

$$\begin{aligned}
\tfrac{d}{d\varepsilon}\big|_{\varepsilon=0}\mathcal{I}(\Omega_\varepsilon) &= \frac{\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\text{length}(\partial\Omega_\varepsilon)}{\text{area}(\Omega_0)^{\frac{1}{2}}} - \frac{1}{2}\frac{\text{length}(\partial\Omega_0)\frac{d}{d\varepsilon}\big|_{\varepsilon=0}\text{area}(\Omega_\varepsilon)}{\text{area}(\Omega_0)^{\frac{3}{2}}} \\
&\le (<) - \frac{2\pi}{\text{area}(\Omega_0)^{\frac{1}{2}}} + \frac{1}{2}\frac{\text{length}(\partial\Omega_0)^2}{\text{area}(\Omega_0)^{\frac{3}{2}}} \\
&= \frac{1}{2}\frac{-4\pi + \mathcal{I}(\partial\Omega_0)^2}{\text{area}(\Omega_0)^{\frac{1}{2}}} \le 0,
\end{aligned}$$

since $\mathcal{I}(\Omega_0)^2 \le 4\pi$. This contradicts the minimality of Ω_0 unless the second inequality is an equality, which implies that $K \equiv 0$ in Ω_0.

3. Characterization of the cut value

Let M be a complete Riemannian manifold. Given $p \in M$ and $u \in TM_p$ we define the *cut value* of u as the number

$$t_u := \sup\{t > 0 \ : \ d(\exp_p(tu), p) = t\}.$$

Let $c_u\colon \mathbb{R} \to M$, $c_u(t) := \exp_p(tu)$, be a unit speed geodesic. If the cut value t_u is finite then (at least) one of the following holds for $t = t_u$:

(i) $c_u(t)$ is the first conjugate point of p along $c_u|_{[0,t]}$,

(ii) there exists $v \in TM_p$, $|v| = 1$, $v \neq u$ with $c_u(t) = c_v(t)$.

Conversely, if (i) or (ii) is satisfied for some $t \in (0, \infty)$, then $t_u \le t$.

Solution. For a sequence $t_i \downarrow t_u$, let γ_i be a minimizing unit speed geodesic from p to $c_u(t_i)$ and define $v_i := \gamma_i'(0) \in S^{m-1} \subset TM_p$. By compactness of S^{m-1}, we may assume that $v_i \to v$. By continuity, we get that c_v is a minimizing geodesic from p to $c_u(t_u)$.
If $v \neq u$, we have assertion (ii).

If $u = v$, we will show that (i) holds. As $c_u = c_v$ is minimizing up to $c_u(t_u)$, there are no conjugate points on $c_u|_{[0,t_u]}$ before $c_u(t_u)$. By Lemma 3.11 it suffices to show that $d\exp_p$ is singular at $t_u u$. If not, then there is some neighborhood U of $t_u u$ where $\exp_p$ is a diffeomorphism. By the definition of $\gamma_i = c_{v_i}$, we have $c_u(t_i) = c_{v_i}(t_i')$ for $t_i' \leq t_i$. For i large enough, it holds that $t_i u, t_i' v_i \in U$. Then

$$\exp_p(t_i u) = c_u(t_i) = c_{v_i}(t_i') = \exp_p(t_i' v_i)$$

and therefore $t_i u = t_i' v_i$ and $u = v_i$, which contradicts the definition of t_u.

Conversely, if (i) holds for some t_0, then $t_u \leq t_0$ since a geodesic does not minimize distance after the first conjugate point.

Suppose now that (ii) holds. Let U be a uniquely geodesic neighborhood of $c_u(t_0) = c_v(t_0)$, take $\varepsilon > 0$ such that $c_u(t_0 + \varepsilon), c_v(t_0 - \varepsilon) \in U$ and let σ be the unique geodesic from $c_v(t_0 - \varepsilon)$ to $c_u(t_0 + \varepsilon)$. Then either $L(\sigma) < 2\varepsilon$, and then then $c_v|_{[0,t_0-\varepsilon]} \cup \sigma$ is a path from p to $c_u(t_0 + \varepsilon)$ which is strictly shorter that $c_u|_{[0,t_0+\epsilon]}$ and therefore $t_u \leq t_0 + \varepsilon$. Or $L(\sigma) = 2\varepsilon$, but then $c_u(t_0 + \varepsilon) = c_v(t_0 - \varepsilon)$ and if $c|_{[0,t_0+\varepsilon]}$ were minimizing we would have

$$t_0 + \varepsilon = L\big(c|_{[0,t_0+\varepsilon]}\big) \leq L\big(\gamma|_{[0,t_0-\varepsilon]}\big) = t_0 - \varepsilon,$$

a contradiction. Thus $t_u \leq t_0 + \varepsilon$, and as this holds for all ε small enough the result follows.