Solutions 2

1. Basic properties of the Lie bracket

Let M be a smooth manifold, X , Y and Z belong to $\Gamma(TM)$, and f, g belong to $C^\infty(M)$

a) Show that the Lie bracket $[\,\cdot\,,\,\cdot\,]$ is bilinear and satisfies:

- $[Y, X] = -[X, Y]$
- $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$
- $[X, [Y, Z]] + [Y, [Z, X] + [Z, [X, Y]] = 0$

b) Show that in a chart (φ, U), if $X = \sum X^i \frac{\partial}{\partial \varphi^i}$ and $Y = \sum Y^j \frac{\partial}{\partial \varphi^j}$, we have

$$[X, Y]|_U = \sum_i \left(\sum_j X^j \frac{\partial Y^i}{\partial \varphi^j} - Y^j \frac{\partial X^i}{\partial \varphi^j} \right) \frac{\partial}{\partial \varphi^i}$$

Solution. a) Note first that $[X, Y]$ is characterised by how it acts on $C^\infty(M)$ functions, that is $[X, Y]f = X(Y(f)) - Y(X(f))$ for all $f \in C^\infty(M)$. It is clear from this characterisation that $[\,\cdot\,,\,\cdot\,]$ is bilinear and antisymmetric. To see the other identities we note that, on the one hand, for all $h \in C^\infty(M)$ we have

$$\begin{aligned}
[fX, gY](h) &= fX(gY(h)) - gY(fX(h)) \\
&= fX(g)Y(h) + fgX(Y(h)) - gY(f)X(h) - gfY(X(h)) \\
&= \big(fg[X, Y] + fX(g)Y - gY(f)X\big)(h).
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\left(\textstyle\sum_{\text{cyclic}}[X, [Y, Z]]\right)(h) = \\
&= \textstyle\sum_{\text{cyclic}}\big(X(Y(Z(h))) - X(Z(Y(h))) - Y(Z(X(h))) + Z(Y(X(h)))\big) \\
&= 0.
\end{aligned}$$

b) As shown in the lecture, thanks to the commutation of second partial derivatives we have $\left[\frac{\partial}{\partial \varphi^i}, \frac{\partial}{\partial \varphi^j}\right] = 0$ for all i, j. Hence using the properties shown in a)

$$
\begin{aligned}
[X,Y]|_U &= \sum_{i,j}\left[X^i\frac{\partial}{\partial\varphi^i}, Y^j\frac{\partial}{\partial\varphi^j}\right]\\
&= \sum_{i,j}\left(X^i\frac{\partial Y^j}{\partial\varphi^i}\frac{\partial}{\partial\varphi^j} - Y^j\frac{\partial X^i}{\partial\varphi^j}\frac{\partial}{\partial\varphi^i}\right)\\
&= \sum_{i}\left(\sum_j X^j\frac{\partial Y^i}{\partial\varphi^j} - Y^j\frac{\partial X^i}{\partial\varphi^j}\right)\frac{\partial}{\partial\varphi^i}.
\end{aligned}
$$

2. The Levi-Civita connection on a submanifold

Let $(\bar{M},\bar{g})$ be a Riemannian manifold with Levi-Civita connection $\bar{D}$, and let M be a submanifold of $\bar{M}$, equipped with the induced metric $g := i^*\bar{g}$, where $i\colon M \to \bar{M}$ is the inclusion map.

Show that the Levi-Civita connection D of (M,g) satisfies $D_XY = (\bar{D}_XY)^{\mathrm{T}}$ for all $X,Y\in\Gamma(TM)$, where the superscript T denotes the component tangential to M and $\bar{D}_XY$ is defined(!) as $\bar{D}_XY := \bar{D}_{\bar{X}}\bar{Y}$ for any extensions $\bar{X},\bar{Y}\in\Gamma(T\bar{M})$ of X,Y.

Solution. As we have seen in the lecture (Remark 1.7), that $(\bar{D}_{\bar{X}}\bar{Y})_p$ only depends on $\bar{X}_p$ and $\bar{Y}\circ c$, where $c\colon(-\epsilon,\epsilon)\to\bar{M}$ is a curve with $\dot{c}(0)=\bar{X}$. Hence $\bar{D}_XY$ is independent of the choice of the extensions $\bar{X}$ and $\bar{Y}$.

Clearly, $(\bar{D}_XY)^{\mathrm{T}}$ defines a linear connection. It remains to prove that this connection is compatible with g and torsion-free. For $X,Y,Z\in TM$, we have

$$
\begin{aligned}
Zg(X,Y) &= \bar{Z}\bar{g}(\bar{X},\bar{Y}) = \bar{g}(\bar{D}_{\bar{Z}}\bar{X},\bar{Y}) + \bar{g}(\bar{X},\bar{D}_{\bar{Z}}\bar{Y})\\
&= \bar{g}((\bar{D}_ZX)^{\mathrm{T}},\bar{Y}) + \bar{g}(\bar{X},(\bar{D}_ZY)^{\mathrm{T}}) = g((\bar{D}_ZX)^{\mathrm{T}},Y) + g(X,(\bar{D}_ZY)^{\mathrm{T}})
\end{aligned}
$$

and

$$
(\bar{D}_XY) - (\bar{D}_YX) = (\bar{D}_{\bar{X}}\bar{Y})^{\mathrm{T}} - (\bar{D}_{\bar{Y}}\bar{X})^{\mathrm{T}} = [\bar{X},\bar{Y}]^{\mathrm{T}} = [X,Y].
$$

3. Gradient and Hessian form

Let (M, g) be a Riemannian manifold, D the Levi-Civita connection and $f\colon M \to \mathbb{R}$ a smooth function on M.

a) The *gradient* $\mathrm{grad} f \in \Gamma(TM)$ is defined by

$$df(X) = g(\mathrm{grad} f, X), \quad \forall X \in \Gamma(TM).$$

Compute $\mathrm{grad} f$ in local coordinates.

b) The *Hessian form* $\mathrm{Hess}(f) \in \Gamma(T_{0,2}M)$ is defined by

$$\mathrm{Hess}(f)(X, Y) = g(D_X \mathrm{grad} f, Y), \quad \forall X, Y \in \Gamma(TM).$$

Prove that $\mathrm{Hess}(f)$ is symmetric and compute $\mathrm{Hess}(f)$ in local coordinates.

Solution. a) For a chart (ϕ, U), let $A_i := \frac{\partial}{\partial \phi^i}$ and $\mathrm{grad} f = \sum_i Y^i A_i$. Then we have

$$\begin{aligned} f_j := \tfrac{\partial}{\partial \phi^j}(f) &= df\,(A_j) = g\,(\mathrm{grad} f, A_j) \\ &= g\left(\sum_i Y^i A_i, A_j\right) = \sum_i Y^i g\,(A_i, A_j) = \sum_i Y^i g_{ij} \end{aligned}$$

Hence we get $Y^i = \sum_j f_j g^{ji}$ and thus $\mathrm{grad} f = \sum_{i,j} g^{ji} f_j A_i$.

b) First, we use that D is compatible with g. We get

$$\begin{aligned} \mathrm{Hess}(f)(X, Y) &= g(D_X \mathrm{grad} f, Y) = Xg(\mathrm{grad} f, Y) - g(\mathrm{grad} f, D_X Y) \\ &= X(Y(f)) - (D_X Y)(f) \end{aligned}$$

Since the Levi-Civita connection D is torsion free, it follows

$$\begin{aligned} \mathrm{Hess}(f)(X, Y) &= X(Y(f)) - (D_X Y)(f) + T(X, Y)(f) \\ &= Y(X(f)) - (D_Y X)(f) = \mathrm{Hess}(f)(Y, X), \end{aligned}$$

i.e. $\mathrm{Hess}(f)$ is symmetric. Furthermore, we get in local coordinates

$$\mathrm{Hess}(f)_{ij} = \mathrm{Hess}(f)(A_i, A_j) = A_i(A_j(f)) - (D_{A_i} A_j)(f) = f_{ij} - \sum_k \Gamma_{ij}^k f_k.$$