Solutions 3

1. Motivation of the geodesic equation

Let (M,g) be a compact Riemannian manifold and $c:[a,b]\to M$ a smooth curve parametrised by the arc length. Suppose that $c([a,b])$ is covered by one chart (ϕ,U). Construct m smooth variations $\gamma_\ell:(-\epsilon,\epsilon)\times[a,b]\to M$ such that the associated variation vector fields along c , $V_{\ell,0}:=\big((\gamma_\ell)_*\frac{\partial}{\partial s}\big)(0,\,\cdot\,)\in\Gamma(c^*TM)$, satisfy that $\{V_{\ell,0}(t)\}_{1\le\ell\le m}$ is a basis of $TM_{c(t)}$ for all $t\in[a,b]$. Deduce that any smooth length-minimising curve is a geodesic (from the first variation of arc length [Theorem 1.15 in Prof. Lang's notes]).

Solution. Define

$$\gamma_\ell := \phi^{-1}(\phi\circ c + se_\ell),$$

where $e_\ell=\frac{\partial}{\partial x^\ell}$ is the ℓ-th vector of the standard basis of $\mathbb{R}^m$. Then

$$V_{\ell,0}(t)=d(\phi^{-1})_{(\phi\circ c)(t)}\big(e_\ell\big).$$

Note that $d(\phi^{-1})_{(\phi(p)(t)}\big(e_\ell\big)$ is a basis of TM_p for all $p\in U$ (since ϕ is a chart). Therefore, we constructed variations satisfying the desired properties. Using the formula of the first variation of the arc length and minimality of c we obtain, for any variation with associated vector field V_0 vanishing at a and b:

$$0=\tfrac{d}{ds}\big|_{s=0}L(\gamma_s)=-\int_a^b\langle V_0(t),\tfrac{D}{dt}\dot c(t)\rangle dt.$$

Suppose now by contradiction that the (continuous) vector field $\frac{D}{dt}\dot c(t)$ along c was not zero at some $t_0\in(a,b)$. Then for some $\delta>0$ and $\ell\in 1,\dots,m$ we would have $\langle V_{\ell,0}(t),\frac{D}{dt}\dot c(t)\rangle\neq 0$ (say >0) for all $t\in(t_0-\delta,t_0+\delta)$. Define

$$\gamma=\phi^{-1}(\phi\circ c+s\psi(t)e_\ell),$$

where $\psi\ge 0$ compactly supported in $(t_0-\delta,t_0+\delta)$ and not identically zero. The associated variation vector field along c is

$$V_0(t)=d(\phi^{-1})_{(\phi\circ c)(t)}\big(\psi(t)e_\ell\big)=\psi(t)d(\phi^{-1})_{(\phi\circ c)(t)}\big(e_\ell\big)=\psi(t)V_{0,\ell}.$$

Hence we obtain

$$0=-\int_a^b\langle V_0(t),\tfrac{D}{dt}\dot c(t)\rangle dt=-\int_{t_0-\delta}^{t_0+\delta}\psi\langle V_{0_\ell}(t),\tfrac{D}{dt}\dot c(t)\rangle dt<0;$$

a contradiction.

2. Existence of closed geodesics

Let (M, g) be a compact Riemannian manifold and $c_0 \colon S^1 \to M$ a continuous closed curve. The purpose of this exercise is to show that in the family of all continuous and piece-wise C^1 curves $c \colon S^1 \to M$ which are homotopic to c_0, there is a shortest one and it is a geodesic.

a) Show that c_0 is homotopic to a piece-wise C^1-curve c_1 with finite length.

b) Let $L := \inf_c L(c)$ be the infimum over all piece-wise C^1 curves $c \colon S^1 \to M$ homotopic to c_0 and consider a minimizing sequence $(c_n \colon S^1 \to M)_n$ with $\lim_n L(c_n) = L$. Use compactness of M to construct a piece-wise C^1-curve $c \colon S^1 \to M$ with length L.

   Hint. Cover M with simply connected balls with the property that every two points in a ball are joined by a unique distance minimizing geodesic.

c) Conclude by showing that c is homotopic to c_0 and a geodesic.

Solution. a) Let us first prove that c_0 is homotopic to a piece-wise C^1-curve c_1. To this aim, we split c_0 into finitely many paths $\gamma_i \colon [0, 1] \to M$ such that $\gamma_i(1) = \gamma_{i+1}(0)$, $\gamma_n(1) = \gamma_1(0)$ and γ_i is contained in a charts $\{(\phi_i, U_i)\}_{i=1}^n$ with U_i simply-connected. Then γ_i is homotopic (relative to the endpoints) to a C^1-curve $\widetilde{\gamma}_i$ and by connecting the $\widetilde{\gamma}_i$'s we get a piece-wise C^1-curve c_1 which is homotopic to c_0. Then c_1 has finite length $L(c_1)$.

b) Let $L := \inf_c L(c) < \infty$ be the infimum over all curves $c \colon S^1 \to M$ which are piece-wise C^1 and homotopic to c_0 and consider a minimizing sequence, i.e. a sequence $(c_n \colon S^1 \to M)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} L(c_n) = L$.

We may assume that the curves $c_n \colon [0, 1] \to M$ are parametrized proportional to arclength, i.e. $L(c_n|_{[a,b]}) = |b - a| \cdot L(c_n)$.

As M is compact, there is some $r > 0$ and points $q_q, \dots, q_n \in M$ such that the balls $B(q_1, r), \dots, B(q_n, r)$ cover M, for all $q, q' \in B(q_i, 3r)$ there is a unique distance minimizing geodesic joining q to q' of length $< 6r$ and the balls $B(q_i, 6r)$ are simply connected.

Fix some $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{r}{L}$ and define $t_k := \frac{k}{N}$ for $k = 0, \dots, N$. Consider now the sequences $(c_n(t_k))_{n\in\mathbb{N}}$. By compactness of M, we may assume (by possibly passing to subsequences) that $c_n(t_k) \to p_k$ for each $k = 0, \dots, N$. Therefore

$$d(p_k, p_{k+1}) \leq \limsup_{n\to\infty} d(c_n(t_k), c_n(t_{k+1})) \leq \limsup_{n\to\infty} \tfrac{1}{N} L(c_n) < r.$$

Take $q \in \{q_1, \dots, q_n\}$ such that $p_k \in B(q,r)$, then $p_{k+1} \subset B(q,3r)$ and therefore we can define a continuous, piece-wise C^1-curve $c\colon [0,1] \to M$ by concatenating the unique distance minimizing geodesics between p_k and p_{k+1}.

For the length of c we have

$$L(c) = \sum_{k=0}^{N-1} L\left(c|_{[t_k,t_{k+1}]}\right) = \sum_{k=0}^{N-1} d(p_k,p_{k+1}) \leq N \limsup_{n\to\infty} \tfrac{1}{N} L(c_n) = L.$$

c) It remains to prove that c is homotopic to c_0. Observe that for n large enough, we have $c([t_k,t_{k+1}]), c_n([t_k,t_{k+1}]) \subset B(q,3r)$.

Since $B(q,6r)$ is simply-connected there is a homotopy from $c_n|_{[\frac{k}{N},\frac{k+1}{N}]}$ to $c|_{[\frac{k}{N},\frac{k+1}{N}]}$ with the endpoints following the unique geodesics from $c_n(t_k)$ to p_k and from $c_n(t_{k+1})$ to p_{k+1}, respectively. Combining this homotopies, we get a homotopy from c_n to c.

Observe that c is locally length minimizing and hence is a geodesic.

3. Metric and Riemannian isometries

Let (M,g) and $(\bar{M},\bar{g})$ be two connected Riemannian manifolds with induced distance functions d and $\bar{d}$, respectively. Further, let $f\colon (M,d) \to (\bar{M},\bar{d})$ be an isometry of metric spaces, i.e. f is surjective and for all $p,p' \in M$ we have $\bar{d}(f(p),f(p')) = d(p,p')$.

a) Prove that for every geodesic γ in M, $\bar{\gamma} := f \circ \gamma$ is a geodesic in N.

b) Let $p \in M$. Define $F\colon TM_p \to T\bar{M}_{f(p)}$ with

$$F(X) := \left.\frac{d}{dt}\right|_{t=0} f \circ \gamma_X(t),$$

where γ_X is the geodesic with $\gamma_X(0) = p$ and $\dot{\gamma}(0) = X$. Show that F is surjective and satisfies $F(cX) = cF(X)$ for all $X \in TM_p$ and $c \in \mathbb{R}$.

c) Conclude that F is an isometry by proving $\|F(X)\| = \|X\|$.

d) Prove that F is linear and conclude that f is smooth in a neighborhood of p.

e) Prove that f is a diffeomorphism for which $f^*\bar{g} = g$ holds.

Solution. a) As the property of being a geodesic is local, we may assume that both $\gamma\colon [0,L] \to M$ and $f \circ \gamma\colon [0,L] \to \bar{M}$ are contained in an open set

$U \subset M$ and $\bar{U} \subset \bar{M}$, respectively, such that points in U and $\bar{U}$ are connected by a unique distance minimizing geodesic. Then there is a unique geodesic β from $\bar{\gamma}(0)$ to $\bar{\gamma}(L)$. We claim that $\bar{\gamma}$ and β coincide.

In the following all geodesics are parametrized by arclength. For $t \in [0, L]$ there are geodesics β_1 from $\bar{\gamma}(0)$ to $\bar{\gamma}(t)$ and β_2 from $\bar{\gamma}(t)$ to $\bar{\gamma}(L)$. Concatenating β_1 and β_2, we get some piece-wise C^1-curve from $\bar{\gamma}(0)$ to $\bar{\gamma}(L)$ with length

$$\begin{aligned} L(\beta_1\beta_2) &= L(\beta_1) + L(\beta_2) \\ &= \bar{d}(\bar{\gamma}(0), \bar{\gamma}(t)) + \bar{d}(\bar{\gamma}(t), \bar{\gamma}(L)) \\ &= d(\gamma(0), \gamma(t)) + d(\gamma(t), \gamma(L)) \\ &= d(\gamma(0), \gamma(L)) = \bar{d}(\bar{\gamma}(0), \bar{\gamma}(L)) = L(\beta). \end{aligned}$$

Hence, by uniqueness of the geodesic from $\bar{\gamma}(0)$ to $\bar{\gamma}(L)$, $\beta_1\beta_2$ and β coincide, i.e. $\bar{\gamma}(t) = \beta(t)$.

b) Observe that f is bijective and its inverse f^{-1} is also is an isometry of metric spaces.

First, we prove that F is surjective. Let $Y \in T\bar{M}_{f(p)}$ and $\bar{\gamma}$ the geodesic through $f(p)$ with $\dot{\bar{\gamma}}(0) = Y$. Then $Y = F(X)$ for $X := \frac{d}{dt}\big|_{t=0} f^{-1} \circ \bar{\gamma}(t)$.

From $\gamma_{cX}(t) = \gamma_X(ct)$ it follows that

$$F(cX) = \frac{d}{dt}\bigg|_{t=0} f \circ \gamma_X(ct) = cF(X).$$

c) For $\epsilon > 0$ small enough, we have that $\gamma_X(\epsilon)$ and $f \circ \gamma_X(\epsilon)$ are contained in a normal neighborhood of p and $f(p)$, respectively. Hence we get

$$\epsilon\|X\| = d(p, \gamma_X(\epsilon)) = \bar{d}(f(p), f \circ \gamma_X(\epsilon)) = \epsilon\|F(X)\|.$$

We now claim that for $X, Y \in TM_p$ with $\|X\| = \|Y\| = 1$ and α such that $\cos\alpha = g_p(X, Y)$ we have

$$\sin\frac{1}{2}\alpha = \lim_{s\to 0} \frac{1}{2s} d(\gamma_X(s), \gamma_Y(s)),$$

and a similar formula for $\bar{X}, \bar{Y} \in T\bar{M}_f(p)$ with $\|\bar{X}\| = \|\bar{Y}\| = 1$.

Assuming the claim for the moment, we now prove that

$$g_p(X, Y) = \bar{g}_{f(p)}(F(X), F(Y)).$$

for all $X, Y \in TM_p$.

Note first that since $F(cX) = cF(X)$, we can assume that $\|X\| = \|Y\| = 1$, then also $\|F(X)\| = \|F(Y)\| = 1$. So by the claim and the fact that

f is a distance preserving map we have for $\cos\alpha = g_p(X,Y)$ and $\cos\alpha' = \bar{g}_{f(p)}(F(X), F(Y))$

$$\sin\frac{1}{2}\alpha = \sin\frac{1}{2}\alpha'.$$

Therefore

$$g_p(X,Y) = \cos\alpha = 1 - 2\sin^2\frac{1}{2}\alpha = 1 - 2\sin^2\frac{1}{2}\alpha' = \bar{g}_{f(p)}(F(X), F(Y)).$$

d)For all $X, Y, Z \in TM_p$ and $c \in \mathbb{R}$, we have

$$\begin{aligned}\bar{g}_{f(p)}(F(X+cY), F(Z)) &= g_p(X+cY, Z)\\ &= g_p(X,Z) + c g_p(Y,Z)\\ &= \bar{g}_{f(p)}(F(X), F(Z)) + c\bar{g}_{f(p)}(F(Y), F(Z))\\ &= \bar{g}_{f(p)}(F(X) + cF(Y), F(Z))\end{aligned}$$

Hence F is linear and therefore smooth.

If V_p is a neighborhood of $0 \in TM_p$ such that $\exp_p|_{V_p} \colon V_p \to U_p$ is a diffeomorphism, then we have

$$f|_{U_p} = \exp_{f(p)} \circ F \circ (\exp_p|_{V_p})^{-1}.$$

Hence f is smooth as well.

e) The argument above works for all $p \in M$ and also for f^{-1}. Hence f is a diffeomorphism. Furthermore, we have

$$df_p = d(\exp_{f(p)} \circ F \circ \exp_p^{-1}) = F$$

and thus

$$f^*\bar{g}_p(X_p, Y_p) = \bar{g}_{f(p)}(df_p(X_p), df_p(Y_p)) = \bar{g}_{f(p)}(F(X_p), F(Y_p)) = g_p(X_p, Y_p),$$

for all $X, Y \in TM$.

Proof of the claim (sketch). Let $X, Y \in TM_p$ with $\|X\| = \|Y\| = 1$ and let $\alpha = \sphericalangle_0(X,Y)$, that is, $\cos\alpha = g_p(X,Y)$. Consider normal coordinates $(\varphi, B(p,r))$ around p, so that we have $\varphi\colon B(p,r) \to B_r \subset \mathbb{R}^n$ and define $c_X := \varphi \circ \gamma_X$ and $c_Y := \varphi \circ \gamma_Y$, two curves in B_r.

On B_r we can consider two different metrics. The Euclidean metric g_E and the pull-back metric $h := (\varphi^{-1})^* g$.

Note that $h_0(c_X'(0), c_Y'(0)) = g_p(X,Y)$ and by Lemma 1.19 $h_0 = (g_E)_0$, so $(g_E)_0(c_X'(0), c_Y'(0)) = g_p(X,Y)$. We are now in a completely Euclidean setting.

Suppose by contradiction that $\limsup_{s\to 0} \frac{1}{2s} d(\gamma_X(s), \gamma_Y(s)) > \sin\frac{1}{2}\alpha$ and take $c > 1$ such that

$$\limsup_{s\to 0} \tfrac{1}{2s} d(\gamma_X(s), \gamma_Y(s)) > c \sin\tfrac{1}{2}\alpha.$$

Now, take r small enough such that $c^{-1} \cdot g_E < h < c \cdot g_E$ on $B_r \subset \mathbb{R}^n$, and therefore

$$c^{-1} \cdot d_E < d_h < c \cdot d_E,$$

where d_h denotes the distance function induced by the metric h. This implies that for s small enough

$$\frac{1}{2s} d_E(c_X(s), c_Y(s)) > c \sin\frac{1}{2}\alpha,$$

which is not true. The other inequality follows similarly. □