Solutions 4

1. Applications of Hopf-Rinow

(a) Let (M, g) be a *homogeneous Riemannian manifold*, i.e. the isometry group of M acts transitively on M. Prove that M is geodesically complete.

(b) Show that if (M, g) is a complete non-compact Riemannian manifold then there exist a ray emanating from any given $p \in M$, that is, a geodesic $c : [0, +\infty) \to M$ such that $c_v(0) = p$ and $\mathrm{dist}(p, c_v(t)) = t$ for all $t \geq 0$.

Solution. (a) Let $p \in M$. Pick $r > 0$ such that $\exp_p$ is defined on $B(0, r) \subset TM_p$. Let $v \in T_pM$ be a tangent vector and let (α_v, ω_v) be the maximal interval, where the geodesic c_v satisfying $c_v(0) = p$ and $\dot{c}_v(0) = v$ is defined. We need to show that $(\alpha_v, \omega_v) = (-\infty, \infty)$. Suppose that $\omega_v < \infty$. Let $0 < \epsilon < r$. Consider $q = c_v(\omega_v - \epsilon) \in M$. By assumption, there exists an isometry Φ of M such that $\Phi(p) = q$. Put $w := D\Phi_q^{-1}(\dot{c}_v(\omega_v - \epsilon)) \in T_pM$ and let c_w be the associated geodesic. Then $\Phi \circ c_w$ is a geodesic starting at q that extends c_v to $(\alpha_v, \omega_v + r - \epsilon)$. This is a contradiction to the maximality of ω_v. Hence $\omega_v = \infty$. Similarly one shows $\alpha_v = -\infty$.

This shows that $\exp_p(tv)$ is defined on $(-\infty, \infty)$ and therefore M is geodesically complete.

(b) Let $p \in M$. Pick $r > 0$ such that $\exp_p$ is a diffeomorphism in a neighborhood of on $\overline{B}(0, r) \subset TM_p$. Since M is complete and non-compact, there exist a sequence p_k such that $t_k := \mathrm{dist}(p, p_k) \to \infty$.

For all k let $v_k \in TM_p$ be a unit vector such that $c_{v_k} : [0, t_k] \to M$ is a minimizing geodesic joining p and p_k. (Such minimizing geodesic exists by Hopf-Rinow). In particular $\mathrm{dist}(p, c_{v_k}(t)) = t$ for all $t \in [0, t_k]$. Take a partial subsequence $v_k \to v$. Then c_v satisfies, by continuity of $v \mapsto c_v(t)$, $\mathrm{dist}(p, c_v(t)) = t$ for every $t > 0$.

Remark. By the Theorem of Hopf-Rinow this implies that M is complete.

2. Ricci curvature

Let (M, g) be a 3-dimensional Riemannian manifold. Show the following:

a) The Ricci curvature ric uniquely determines the Riemannian curvature tensor R.

b) If M is an Einstein manifold, that is, a Riemannian manifold (M, g) with ric $=$ kg for some $k \in \mathbb{R}$, then the sectional curvature sec is constant.

Solution. a) In the following, let e_1, e_2, e_3 be an orthonormal basis of TM_p. First, note that $R_{iijk} = R_{jkii} = 0$ by the symmetry properties of R.

We denote the components of ric by R_{ij}. Then, for $\{i, j, k\} = \{1, 2, 3\}$, we have

$$\begin{aligned} R_{ii} &= R_{iiii} + R_{jiji} + R_{kiki} = R_{ijij} + R_{ikik}, \\ R_{ij} &= R_{iiij} + R_{jijj} + R_{kikj} = R_{ikjk} \end{aligned}$$

and therefore, we get

$$\begin{aligned} 2R_{ijij} &= R_{ii} + R_{jj} - R_{kk}, \\ R_{ikjk} &= R_{ij}. \end{aligned}$$

Observe now, that we can compute all other components of R by symmetry properties. Hence R is uniquely determined by ric.

b) Let e_1, e_2 be a orthonormal basis of $E \subset TM_p$ and choose e_3 such that e_1, e_2, e_3 is an orthonormal basis of TM_p. Then we have

$$2 \sec_p(E) = 2R_{1212} = R_{11} + R_{22} - R_{33} = k + k - k = k$$

and hence $\sec_p(E) = \frac{k}{2}$.

3. Constant sectional curvature

Let (M, g) be a Riemannian manifold with constant sectional curvature $\sec(E) = \kappa \in \mathbb{R}$ for all $E \in G_2(TM)$. Show that

$$R(X, Y)W = \kappa \left(g(Y, W)X - g(X, W)Y \right).$$

Solution. As the sectional curvature is constant, we have

$$R(X, Y, X, Y) = \kappa \left(g(X, X)g(Y, Y) - g(X, Y)g(X, Y) \right)$$

for all $X, Y \in \Gamma(TM)$. Consider now the $(0, 4)$-tensor T given by

$$T(V, W, X, Y) \coloneqq \kappa \left(g(V, X)g(Y, W) - g(V, Y)g(X, W) \right).$$

Then the $(0, 4)$-tensor $S \coloneqq R - T$ has the following symmetry properties:

1. $S(V, W, X, Y) = -S(V, W, Y, X)$,

2. $S(V,W,X,Y) + S(V,Y,W,X) + S(V,X,Y,W) = 0$,
3. $S(V,W,X,Y) = S(X,Y,V,W)$,
4. $S(X,Y,X,Y) = 0$.

The first three properties hold for R and T, while the last one was already observed above. Our goal is now to show that $S \equiv 0$.

For all $A, B, C, D \in \Gamma(TM)$, we have by 3. and 4.

$$\begin{aligned} 0 &= S(A, B+D, A, B+D) \\ &= S(A,B,A,B) + S(A,B,A,D) + S(A,D,A,B) + S(A,D,A,D) \\ &= 2S(A,B,A,D) \end{aligned}$$

and

$$\begin{aligned} 0 &= S(A+C, B, A+C, D) \\ &= S(A,B,A,D) + S(A,B,C,D) + S(C,B,A,D) + S(C,B,C,D) \\ &= S(A,B,C,D) + S(A,D,C,B). \end{aligned}$$

Finally, we get

$$\begin{aligned} 3S(V,W,X,Y) &= S(V,W,X,Y) - S(V,Y,X,W) - S(V,W,Y,X) \\ &= S(V,W,X,Y) + S(V,Y,W,X) + S(V,X,Y,W) = 0, \end{aligned}$$

for all $V, W, X, Y \in \Gamma(TM)$.