Solutions 5

1. Divergence and Laplacian

Let (M, g) be a Riemannian manifold with Levi-Civita connection D. The *divergence* $\operatorname{div}(Y)$ of a vector field $Y \in \Gamma(TM)$ is the contraction of the $(1,1)$-tensor field $DY \colon X \mapsto D_X Y$ and the *Laplacian* $\Delta \colon C^\infty(M) \to C^\infty(M)$ is defined by $\Delta f := \operatorname{div}(\operatorname{grad} f)$. Show that:

a) $\operatorname{div}(fY) = Y(f) + f \operatorname{div} Y$,

b) $\Delta(fg) = f\Delta g + g\Delta f + 2\langle \operatorname{grad} f, \operatorname{grad} g\rangle$,

c) Compute Δf in local coordinates.

Solution. a) Let $p \in M$ and let $e_1, \dots, e_n$ be a orthonormal basis of TM_p. Then we have

$$\begin{aligned}
\operatorname{div}_p(fY) &= \sum_{i=1}^n \langle D_{e_i}(fY), e_i\rangle \\
&= \sum_{i=1}^n \langle e_i(f) Y_p + f(p) D_{e_i} Y, e_i\rangle \\
&= \sum_{i=1}^n e_i(f)\langle Y_p, e_i\rangle + \sum_{i=1}^n f(p)\langle D_{e_i} Y, e_i\rangle \\
&= Y_p(f) + f(p)\operatorname{div}_p(Y) \\
&= (Y(f) + f\operatorname{div}(Y))(p)
\end{aligned}$$

and hence $\operatorname{div}(fY) = Y(f) + f \operatorname{div} Y$.

b) First, recall the definition of $\operatorname{grad} f$, i.e. $X(f) = \langle \operatorname{grad} f, X\rangle$ and note that

$$\langle \operatorname{grad}(fg), X\rangle = X(fg) = X(f)g + fX(g) = \langle \operatorname{grad}(f)g + f\operatorname{grad}(g), X\rangle,$$

for all $X \in \Gamma(M)$ and thus $\operatorname{grad}(fg) = \operatorname{grad}(f)g + f\operatorname{grad}(g)$.

Therefore, we get

$$\begin{aligned}
\Delta(fg) &= \operatorname{div}(\operatorname{grad}(fg)) \\
&= \operatorname{div}(\operatorname{grad}(f)g + f\operatorname{grad}(g)) \\
&= \operatorname{div}(\operatorname{grad} f)g + \operatorname{grad}(f)(g) + f\operatorname{div}(\operatorname{grad}(g)) + \operatorname{grad}(g)(f) \\
&= \Delta(f)g + f\Delta(g) + 2\langle \operatorname{grad} f, \operatorname{grad} g\rangle.
\end{aligned}$$

c) In the following, we use Einstein notation. One can show that in local coordinates

$$\begin{aligned} \operatorname{grad} f &= g^{ik} f_i A_k, \\ D_{A_i} Y &= (A_i(Y^k) + Y^j \Gamma_{ij}^k) A_k. \end{aligned}$$

Therefore, we have

$$\operatorname{div}(Y) = A_k(Y^k) + Y^j \Gamma_{kj}^k.$$

Hence, we get

$$\begin{aligned} \Delta f &= \operatorname{div}(\operatorname{grad} f) \\ &= A_k(g^{ik} f_i) + g^{ij} f_i \Gamma_{kj}^k. \end{aligned}$$

With $G := \det(g_{ij})$, we can simplify this as follows. We have by the Jacobi formula and the chain rule

$$\Gamma_{kj}^k = \frac{1}{2} g^{kl} A_j(g_{kl}) = \frac{1}{2G} A_j(G) = \frac{1}{\sqrt{G}} A_j(\sqrt{G})$$

and therefore

$$\begin{aligned} \Delta f &= A_j(g^{ij} f_i) + g^{ij} f_i \frac{1}{\sqrt{G}} A_j(\sqrt{G}) \\ &= \frac{1}{\sqrt{G}} A_j \left(\sqrt{G} g^{ij} f_i \right). \end{aligned}$$

2. Codazzi equation

Let $M \subset \bar{M}$ be a submanifold of the Riemannian manifold $(\bar{M}, \bar{g})$. For the second fundamental form h of M, we define

$$(D_X^\perp h)(Y, W) := (\bar{D}_X(h(Y, W)))^\perp - h(D_X Y, W) - h(Y, D_X W),$$

where $W, X, Y \in \Gamma(TM)$. Show that the Codazzi equation

$$\left(\bar{R}(X, Y)W\right)^\perp = (D_X^\perp h)(Y, W) - (D_Y^\perp h)(X, W)$$

holds for all $W, X, Y \in \Gamma(TM)$.

Solution. As $\bar{D}_Z W = D_Z W + h(Z, W)$ for $W, Z \in \Gamma(TM)$, we get

$$\begin{aligned}
\bar{R}(X,Y)W &= \bar{D}_X \bar{D}_Y W - \bar{D}_Y \bar{D}_X W - \bar{D}_{[X,Y]} W \\
&= \bar{D}_X (D_Y W + h(Y,W)) - \bar{D}_Y (D_X W + h(X,W)) \\
&\quad - (D_{[X,Y]} W + h([X,Y],W)) \\
&= D_X D_Y W + h(X, D_Y W) + \bar{D}_X (h(Y,W)) \\
&\quad - D_Y D_X W - h(Y, D_X W) - \bar{D}_Y (h(X,W)) \\
&\quad - D_{[X,Y]} W - h(D_X Y - D_Y X, W) \\
&= R(X,Y)W \\
&\quad + \bar{D}_X (h(Y,W)) - h(D_X Y, W) - h(Y, D_X W) \\
&\quad - \bar{D}_Y (h(X,W)) + h(D_Y X, W) + h(X, D_Y W).
\end{aligned}$$

Note that we used that D is torsion free, i.e. $[X, Y] = D_X Y - D_Y X$. Now, taking the normal part, we conclude that the Codazzi equation

$$\big(\bar{R}(X,Y)W\big)^{\perp} = (D^{\perp}_X h)(Y,W) - (D^{\perp}_Y h)(X,W)$$

holds.

3. Sectional curvature of submanifolds

Let $(\bar{M}, \bar{g})$ be a Riemannian manifold with sectional curvature $\bar{\sec}$. Let $p \in \bar{M}$ and $L \subset T\bar{M}_p$ an m-dimensional linear subspace.

a) Prove that there is some $r > 0$ such that for the open ball $B_r(0) \subset T\bar{M}_p$, the set $M := \exp_p(L \cap B_r(0))$ is an m-dimensional submanifold of $\bar{M}$.

b) Let g be the induced metric on M and let sec be the sectional curvature of M. Show that for $E \subset TM_p$, we have $\sec_p(E) = \bar{\sec}_p(E)$ and if L is a 2-dimensional subspace, then $\sec \le \bar{\sec}$ on M.

Solution. a) First, we know that there is some $r > 0$ such that the restriction of the exponential map to $B_r(0)$, i.e. $\exp_p|_{B_r(0)} \colon B_r(0) \to \exp_p(B_r(0))$, is a diffeomorphism. Furthermore, note that $L \cap B_r(0)$ is an m-dimensional submanifold of $B_r(0)$ and hence $M = \exp_p(L \cap B_r(0))$ is an m-dimensional submanifold of $\exp_p(B_r(0))$. Finally, as $\exp_p(B_r(0))$ is open in $\bar{M}$, it follows that M is a submanifold of $\bar{M}$ as well.

b) Let $u, v \in E$ be an orthonormal basis of $E \subset TM_p$. Then we have

$$\begin{aligned}
\sec_p(E) &= R_p(u,v,u,v) \\
&= \bar{R}_p(u,v,u,v) + \bar{g}_p(h_p(u,u), h_p(v,v)) - \bar{g}_p(h_p(u,v), h_p(u,v)) \\
&= \bar{\sec}_p(E) + \bar{g}_p(h_p(u,u), h_p(v,v)) - \bar{g}_p(h_p(u,v), h_p(u,v))
\end{aligned}$$

We now prove that $h_p(u,u) = h_p(v,v) = h_p(u,v) = 0$. Extend u, v to an orthonormal basis $e_1 = u, e_2 = v, e_3, \ldots, e_{\bar{m}}$ of $T\bar{M}_p$. Then this basis induces normal coordinates on $\bar{M}$. Hence, we have $\Gamma_{ij}^k(p) = 0$ and thus $(\bar{D}_{e_i} e_j)_p = 0$ for all i, j. In particular this implies that $h_p(u,u) = h_p(v,v) = h_p(u,v) = 0$ as claimed.

Assume now that $L \subset T\bar{M}_p$ is 2-dimensional and let $q := \exp_p(x) \in M$ for $x \in L \cap B_r(0)$. By the above, we may assume that $x \neq 0$.

Define $w := \frac{x}{|x|} \in TM_p$ and let c_w be the unique geodesic with $c(0) = p$ and $\dot{c}(0) = w$. Then we have $q = c_w(|x|)$ and $u := \dot{c}_w(|x|) \in TM_q$ with $|u| = 1$. Furthermore, by Lemma 2.17,we get

$$h_q(u,u) = \left(\frac{\bar{D}}{dt} \dot{c}_w \bigg|_{t=|x|} \right)^{\perp} = 0.$$

To compute the sectional curvature of $E = TM_q$, we extend u to an orthonormal basis u, v of E and get

$$\sec_q(E) = R_q(u,v,u,v) = \bar{R}_q(u,v,u,v) - |h_q(u,v)|^2 \leq \bar{\sec}_q(E)$$

as desired.