Solutions 6

1. Revisiting connections

(a) Fix some manifold M with a connection ∇. Take any $(1,2)$ tensor field F and define

$$\widetilde{\nabla}_X Y := \nabla_X Y + F(X,Y).$$

Show that

- $\widetilde{\nabla}$ is a connection
- For every connection $\hat{\nabla}$ on M there is a unique $(1,2)$ tensor $\hat{F}$ such that $\hat{\nabla} - \nabla = \hat{F}$. Show that in local coordinates $\hat{\Gamma}^k_{ij} - \Gamma^k_{ij} = F^k_{ij}$. Double check that the difference of two Christoffel symbols indeed transforms like a $(1,2)$ tensor field.

(b) Let $\nabla, \widetilde{\nabla}$ be two connections on M and $F(X,Y) := \widetilde{\nabla}_X Y - \nabla_X Y$ be their difference. Show that ∇ and $\widetilde{\nabla}$ have the same geodesics if and only if F is antisymmetric i.e., $F(X,Y) = -F(Y,X)$. Recall that a geodesic for ∇ is a self-parallel curve w.r.t. ∇, this translates in the ODE (with a harmless abuse of notation) $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$.

Conclude that if ∇ and $\widetilde{\nabla}$ have the same geodesics *and* the same torsion then $\nabla = \widetilde{\nabla}$.

Solution. (a) The only non trivial thing to show is that $\widetilde{\nabla}$ satisfies the Leibniz rule:

$$\begin{aligned}\widetilde{\nabla}_X(fY) &= f\nabla_X Y + (Xf)Y + F(X, fY)\\ &= f\widetilde{\nabla}_X Y + (Xf)Y.\end{aligned}$$

Conversely we show that the map $(X,Y) \mapsto \hat{\nabla}_X Y - \nabla_X Y$ is C^∞ bilinear in both arguments. In X is obvious, while in Y is the same computation as above (the term $(Xf)Y$ cancels). If one wants to see this in coordinates remember that

$${}^{(y)}\Gamma^k_{ij} = \frac{\partial y^k}{\partial x^\ell}\frac{\partial x^p}{\partial y^i}\frac{\partial x^q}{\partial y^j}\,{}^{(x)}\Gamma^\ell_{pq} + \frac{\partial y^k}{\partial x^m}\frac{\partial^2 x^m}{\partial y^i \partial y^j},$$

and notice that the "nontensorial" term does not depend on the connection, so

$${}^{(y)}\hat{\Gamma}^k_{ij} - {}^{(y)}\Gamma^k{}_{ij} = \frac{\partial y^k}{\partial x^\ell}\frac{\partial x^p}{\partial y^i}\frac{\partial x^q}{\partial y^j}\left({}^{(x)}\hat{\Gamma}^\ell_{pq} - {}^{(x)}\Gamma^\ell_{pq}\right).$$

In other words $\hat{F}^k_{ij} := \hat{\Gamma}^k_{ij} - \Gamma^k_{ij}$ transforms indeed like a $(1,2)$ tensor under change of coordinates.

(b)In local coordinates $\dot{\gamma} = \dot{\gamma}^k \partial_k$, where $\dot{\gamma}^k = \frac{d\gamma^k}{dt}$, so using the formula for the covariant derivative (and $\ddot{\gamma}^k := \frac{d^2\gamma^k}{dt^2}$) the geodesic ODE becomes:

$$0 = \ddot{\gamma}^k + (\Gamma^k_{ij} \circ \gamma)\dot{\gamma}^i\dot{\gamma}^j = \ddot{\gamma}^k + \frac{1}{2}(\Gamma^k_{ij} \circ \gamma + \Gamma^k_{ji} \circ \gamma)\dot{\gamma}^i\dot{\gamma}^j,$$

where Γ are the Christoffles of ∇. Now, if F is antisymmetric we find

$$\Gamma^k_{ij} + \Gamma^k_{ji} = \widetilde{\Gamma}^k_{ij} + \widetilde{\Gamma}^k_{ji}, \tag{1}$$

so the geodesics of $\widetilde{\nabla}$ and ∇ solve the same ODE.

Conversely if $\widetilde{\nabla}$ and ∇ share the same geodesics pick $p \in M$, $v \in T_pM$ and a vector field X such that $X(p) = v$. Solving the geodesic equation we find $\gamma\colon (-\varepsilon, \varepsilon) \to M$ such that $\gamma(0) = p, \dot{\gamma}(0) = v$ and is self-parallel w.r.t. $\widetilde{\nabla}$ and ∇ , in particular evaluating at $t = 0$

$$0 = \nabla_v X = \widetilde{\nabla}_v X.$$

So we find $F_p(v, v) = (\widetilde{\nabla} - \nabla)_v X = 0$, since $F_p\colon T_pM \times T_pM \to T_pM$ is bilinear and v was arbitrary we find by polarization

$$2F_p(v, w) + 2F_p(w, v) = F_p(v + w, v + w) - F_p(v - w, v - w) = 0 - 0 = 0.$$

Since p was arbitrary we find F is antisymmetric.

Recall that in local coordinates the torsion T of ∇ is given by $T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$. So combining this with (1) we find

$$\begin{cases} \Gamma^k_{ij} - \Gamma^k_{ji} = \widetilde{\Gamma}^k_{ij} - \widetilde{\Gamma}^k_{ji} \\ \Gamma^k_{ij} + \Gamma^k_{ji} = \widetilde{\Gamma}^k_{ij} + \widetilde{\Gamma}^k_{ji} \end{cases}$$

and the unique solution is $\Gamma^k_{ij} = \widetilde{\Gamma}^k_{ij}$.

2. Meaning of the torsion

Consider the manifold $(\mathbb{R}^3, g_{Eucl})$ endowed with the Levi-Civita connection ∇. Define another connection $\widetilde{\nabla}$ by

$$\widetilde{\nabla}_{\partial_i}\partial_j = \varepsilon^k_{ij}\partial_k \quad (\Longleftrightarrow \widetilde{\Gamma}^k_{ij} := \varepsilon^k_{ij}),$$

where ε^k_{ij} is the sign of the permutation $(1, 2, 3) \mapsto (i, j, k)$, and zero otherwise.

- Show that $\widetilde{\nabla}$ is a connection compatible with g, has the same geodesics of ∇, but has non-vanishing torsion. Check that
$$(\widetilde{\nabla}_v X)_p = (\nabla_v X)_p - X_p \times v.$$
- Compute the parallel transport of $v := (1,0,0)$ along $\gamma(t) := (0,0,t)$. This should clarify where is the "torsion".
- Show that, up to a multiplicative $C^\infty(\mathbb{R}^3)$ function in the Christoffels, $\widetilde{\nabla}$ is the unique connection that is g-compatible and has the same geodesics of ∇.

Solution. Notice that a connection is compatible only if its Christoffel symbols satisfy
$$\partial_k g_{ij} = g_{\ell j}\Gamma^\ell_{ki} + g_{\ell i}\Gamma^\ell_{kj}, \tag{2}$$
which in our case ($\partial_k g_{ij} = \partial_k \delta_{ij} = 0$) becomes the identity
$$0 = \varepsilon^j_{ki} + \varepsilon^i_{kj}.$$
Writing the parallel transport equation for $V(t) = V^k(t)\partial_k \in T_{\gamma(t)}\mathbb{R}^3$ we find
$$\dot{V}^k + \varepsilon^k_{ij} V^j \dot{\gamma}^i = \dot{V}^k + \varepsilon^k_{3j} V^j = 0 \text{ for } k = 1,2,3;$$
that becomes the ODE system
$$\begin{cases} \dot{V}^1 - V^2 = 0, \\ \dot{V}^2 + V^1 = 0, \\ \dot{V}^3 = 0, \\ V(0) = (1,0,0) \end{cases}$$
whose solution is $V(t) = (\cos t, -\sin t, 0) \in T_{(0,0,t)}\mathbb{R}^3$.

If $\widetilde{\nabla}$ is g-compatible (see (2)) and has the same geodesics of ∇ (so $\nabla - \widetilde{\nabla}$ is antisymmetric) one gets (the Christoffels of ∇ are identically zero in these coordinates)
$$0 = \widetilde{\Gamma}^j_{ki} + \widetilde{\Gamma}^i_{kj} = \widetilde{\Gamma}^k_{ij} + \widetilde{\Gamma}^k_{ji} \text{ for all } \{i,j,k\} \subseteq \{1,2,3\}.$$
Now playing with the (anti)symmetries one gets
$$\widetilde{\Gamma}^p_{qp} = \widetilde{\Gamma}^p_{pq} = \widetilde{\Gamma}^q_{pp} = 0 \text{ for all } \{p,q\} \subset \{1,2,3\}.$$
So if two indices are the same $\widetilde{\Gamma}$ vanish. When all the three indices are different, instead, we can always move the indices around and find
$$\widetilde{\Gamma}^k_{ij} = \operatorname{sign}(\sigma)\widetilde{\Gamma}^3_{12} \text{ where } \sigma(1) = i, \sigma(2) = j, \sigma(3) = k.$$
This means that $\widetilde{\Gamma}^k_{ij} = \alpha\varepsilon^k_{ij}$ for some $\alpha \in C^\infty(\mathbb{R}^3)$.

3. The sphere

Use Gauss' equations to prove that the sphere of radius $r > 0$,

$$\mathbb{S}_r^n := \{x \in \mathbb{R}^{n+1} \ : \ |x| = 1\}$$

has constant sectional curvatures equal to $1/r^2$.

Solution. Let us denote $\langle \cdot, \cdot \rangle$ the scalar product of $\mathbb{R}^{n+1}$ and (in consistency with the notation used in the lecture) let $\bar{D}$ and D denote respectively the Levy-Civita connections of $\bar{M} = \mathbb{R}^{n+1}$ and of $M = \mathbb{S}_r^n$ (notice that $\bar{D}$ is just standard component-wise differentiation)

Define $N : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ as $N(x) = x/r$ and note that N is a smooth extension of a unit normal vector field to $\mathbb{S}_r$. Clearly given any (smooth extension of a) vector field $Z \in \Gamma(T\mathbb{S}_r)$ we have

$$(\bar{D}_Z N)^i = d\big(\tfrac{x^i}{r}\big)(Z) = \tfrac{1}{r} Z^i \quad \text{that is} \quad \bar{D}_Z N = \tfrac{1}{r} Z$$

We can then compute, for X, Y tangent vector fields to $\mathbb{S}_r$

$$h(X, Y) = (\bar{D}_X Y)^\perp = \langle \bar{D}_X Y, N \rangle N = -\langle Y, D_X N \rangle N = \frac{-1}{r} \langle X, Y \rangle N$$

Hence, using Gauss' equations we obtain, for any pair of perpendicular unit vectores $e_1, e_2 \in TM_x$ we have:

$$\begin{aligned} R(e_1, e_2, e_1, e_2) &= \bar{R}(e_1, e_2, e_1, e_2) + \langle h(e_1, e_1), h(e_2, e_2) \rangle - \langle h(e_1, e_2), h(e_1, e_2) \rangle \\ &= 0 + \frac{1}{r^2} + 0. \end{aligned}$$

In other words all sectional curvatures are equal to $1/r^2$ as claimed.