Solutions 7

1. Jacobi fields in space forms

Let M be a space form with curvature $\kappa \in \mathbb{R}$. Furthermore, let $c\colon \mathbb{R} \to M$ be a geodesic which is parametrized by arc length and $N_0 \in TM_{c(0)}$ with $|N_0| = 1$, $\langle N_0, \dot{c}(0)\rangle = 0$. Determine the Jacobi field Y along c with starting conditions $Y(0) = aN_0$ and $\dot{Y}(0) = bN_0$ for $a, b \in \mathbb{R}$.

Solution. From Exercise 3 in Sheet 4, we know that

$$R(X,Y)W = \kappa\left(g(Y,W)X - g(X,W)Y\right)$$

and thus the Jacobi equation is

$$\ddot{Y} + \kappa\left(g(\dot{c},\dot{c})Y - g(Y,\dot{c})\dot{c}\right) = 0.$$

Consider now an orthonormal basis $\dot{c}(0), N_0, X_0^3, \dots, X_0^n$ of $TM_{c(0)}$ and the corresponding parallel vector fields $\dot{c}, N, X^3, \dots, X^n \in \Gamma(c^*TM)$. We make the Ansatz $Y = f\dot{c} + gN + \sum_{i=3}^n h_i X^i$ and get

$$\dot{Y} = \dot{f}\dot{c} + \dot{g}N + \sum_{i=3}^n \dot{h}_i X^i,$$
$$\ddot{Y} = \ddot{f}\dot{c} + \ddot{g}N + \sum_{i=3}^n \ddot{h}_i X^i.$$

If we insert it into the Jacobi equation, this yields

$$\ddot{f}\dot{c} + \ddot{g}N + \sum_{i=3}^n \ddot{h}_i X^i + \kappa\left(f\dot{c} + gN + \sum_{i=3}^n h_i X^i - f\dot{c}\right) = 0,$$

which is

$$\begin{aligned} \ddot{f} &= 0, & f(0) &= 0, & \dot{f}(0) &= 0,\\ \ddot{g} + \kappa g &= 0, & g(0) &= a, & \dot{g}(0) &= b,\\ \ddot{h}_i + \kappa h_i &= 0, & h_i(0) &= 0, & \dot{h}_i(0) &= 0. \end{aligned}$$

First, we note that this implies $f \equiv 0$ and $h_i \equiv 0$. Hence, we have $Y(t) = g(t)N(t)$. Finally, we distinguish three cases to determine g:

- If $\kappa < 0$, we get $g(t) = a\cosh(\sqrt{-\kappa}t) + b\sinh(\sqrt{-\kappa}t)$,
- if $\kappa = 0$, we get $g(t) = a + bt$, and
- if $\kappa > 0$, we get $g(t) = a\cos(\sqrt{\kappa}t) + b\sin(\sqrt{\kappa}t)$.

2. Trace of a symmetric bilinear form

Let $(V, \langle\cdot,\cdot\rangle)$ be a m-dimensional Euclidean space and let $r\colon V \times V \to \mathbb{R}$ be a symmetric bilinear form. Furthermore, let $S^{m-1} = \{v \in V : |v| = 1\}$ be the unit sphere. Prove that

$$\int_{S^{m-1}} r(v,v)\,\mathrm{dvol}^{S^{m-1}} = \frac{\mathrm{vol}(S^{m-1})}{m}\,\mathrm{tr}(r) = \omega_m\,\mathrm{tr}(r),$$

where $\mathrm{dvol}^{S^{m-1}}$ denotes the induced volume on S^{m-1} and ω_m is the volume of the m-dimensional unit ball.

Solution. Let $e_1, \dots, e_m$ be a ($\langle\cdot,\cdot\rangle$-)orthonormal basis of V such that r is diagonal, that is, $r(v,v) = \sum_{j=1}^m \lambda_j v_j^2$. Moreover, let $\tau_0, \dots, \tau_{m-1} \in \mathrm{SO}(m,\mathbb{R})$ be the isometries defined by $\tau_i e_j \coloneqq e_{i+j \mod m}$. Then we have

$$\begin{aligned}
\int_{S^{m-1}} r(v,v)\ \mathrm{dvol}^{S^{m-1}} &= \frac{1}{m}\sum_{i=0}^{m-1}\int_{S^{m-1}} r(\tau_i v, \tau_i v)\ \mathrm{dvol}^{S^{m-1}} \\
&= \frac{1}{m}\int_{S^{m-1}} \mathrm{tr}(r)\ \mathrm{dvol}^{S^{m-1}} = \frac{\mathrm{vol}(S^{m-1})}{m}\,\mathrm{tr}(r),
\end{aligned}$$

since

$$\begin{aligned}
\sum_{i=0}^{m-1} r(\tau_i v, \tau_i v) &= \sum_{i=0}^{m-1} r\left(\sum_{j=1}^m v_j e_{i+j}, \sum_{j=1}^m v_j e_{i+j}\right) \\
&= \sum_{j=1}^m \left(\sum_{i=0}^{m-1} \lambda_{i+j}\right) v_j^2 \\
&= \mathrm{tr}(r)\sum_{j=1}^m v_j^2 = \mathrm{tr}(r).
\end{aligned}$$

Finally, recall that $\omega_m = \int_0^1 r^{m-1}\mathrm{vol}(S^{m-1})\,dr = \frac{\mathrm{vol}(S^{m-1})}{m}$.

3. Small balls and scalar curvature

Let p be a point in the m-dimensional Riemannian manifold (M, g). The goal is to prove the following Taylor expansion of the volume of the ball $B_r(p)$ as a function of r:

$$\operatorname{vol}(B_r(p)) = \omega_m r^m \left(1 - \frac{1}{6(m+2)} \operatorname{scal}(p) r^2 + \mathcal{O}(r^3)\right).$$

a) Let $v \in TM_p$ with $|v| = 1$, define the geodesic $c(t) := \exp_p(tv)$ and let $e_1 = v, e_2, \ldots, e_m \in TM_p$ be an orthonormal basis. Consider the Jacobi fields Y_i along c with $Y_i(0) = 0$ and $\dot{Y}_i(0) = e_i$ for $i = 2, \ldots m$. Show that the volume distortion factor of $\exp_p$ at tv is given by

$$J(v,t) := \sqrt{\det\left(\langle T_{tv} e_i, T_{tv} e_j \rangle\right)} = t^{-(m-1)} \sqrt{\det\left(\langle Y_i, Y_j \rangle\right)},$$

where $T_{tv} := (d \exp_p)_{tv}$.

b) Let $E_2, \ldots, E_m$ be parallel vector fields along c with $E_i(0) = e_i$. Then the Taylor expansion of Y_i is

$$Y_i(t) = tE_i - \sum_{k=2}^{m} \left(\tfrac{t^3}{6} R(e_i, v, e_k, v) + \mathcal{O}(t^4)\right) E_k.$$

c) Conclude that $J(v,t) = 1 - \frac{t^2}{6} \operatorname{ric}(v,v) + \mathcal{O}(t^4)$.

Hint: Use $\det(I_m + \epsilon A) = 1 + \epsilon \operatorname{tr}(A) + \mathcal{O}(\epsilon^2)$.

d) Prove the above formula for $\operatorname{vol}(B_r(p))$.

Solution. a) As we have seen in the proof of Proposition 3.6, the Jacobi fields Y_i are given as variation vector fields along c of $\alpha_i(s,t) := \exp_p(t(v + se_i))$, i.e.

$$Y_i(t) = \frac{d}{ds}\bigg|_{s=0} \alpha_i(s,t) = T_{tv}(te_i)$$

and therefore $T_{tv} e_i = \frac{1}{t} Y_i(t)$.

Furthermore, we have $\langle T_{tv} v, T_{tv} e_i \rangle = \langle v, e_i \rangle = 0$ by the Gauss Lemma. Then the volume distortion is given by

$$J(v,t) = \sqrt{\det\left(\langle T_{tv} e_i, T_{tv} e_j \rangle\right)} = t^{-(m-1)} \sqrt{\det\left(\langle Y_i, Y_j \rangle\right)}.$$

b) We check that the derivatives coincide. Clearly, we have $Y_i(0) = 0$, $\dot{Y}_i(0) = e_i$ and $\ddot{Y}_i(0) = -R(Y_i(0), \dot{c}(0))\dot{c}(0) = 0$. Furthermore,

$$\begin{aligned}\dddot{Y}_i(0) &= -(D_{\dot{c}}R)(Y_i(0), \dot{c}(0))\dot{c}(0) - R(\dot{Y}_i(0), \dot{c}(0))\dot{c}(0) \\ &= -R(e_i, v)v = -\sum_{k=2}^{m} \langle R(e_i, v)v, e_k\rangle e_k = -\sum_{k=2}^{m} R(e_k, v, e_i, v)e_k.\end{aligned}$$

c) With the above, we get

$$\begin{aligned}\langle Y_i, Y_j\rangle = t^2\langle E_i, E_j\rangle &- \tfrac{t^4}{6}\sum_{k=2}^{m} R(e_i, v, e_k, v)\,\langle E_k, E_j\rangle \\ &- \tfrac{t^4}{6}\sum_{k=2}^{m} R(e_j, v, e_k, v)\,\langle E_i, E_k\rangle + \mathcal{O}(t^5) \\ = t^2\delta_{ij} &- \tfrac{t^4}{3}R(e_i, v, e_j, v) + \mathcal{O}(t^5)\end{aligned}$$

and thus

$$\begin{aligned}J(v,t) &= \sqrt{\det\left(\delta_{ij} - \tfrac{t^2}{3}R(e_i, v, e_j, v) + \mathcal{O}(t^3)\right)} \\ &= \sqrt{1 - \tfrac{t^2}{3}\operatorname{tr}(R(e_i, v, e_j, v)) + \mathcal{O}(t^3)} \\ &= 1 - \tfrac{t^2}{6}\operatorname{ric}(v,v) + \mathcal{O}(t^3).\end{aligned}$$

d)First, we use polar coordinates:

$$\operatorname{vol}(B_r(p)) = \int_{B_r(0)} J(v,t)\ \mathrm{d}x^1 \ldots \mathrm{d}x^m = \int_0^r \int_{S^{m-1}} t^{m-1} J(v,t)\ \mathrm{dvol}^{S^{m-1}}\ \mathrm{d}t.$$

Then, using exercise 2 and the above, we get

$$\begin{aligned}\operatorname{vol}(B_r(p)) &= \int_0^r \int_{S^{m-1}} t^{m-1}(1 - \tfrac{t^2}{6}\operatorname{ric}(v,v) + \mathcal{O}(()t^3))\ \mathrm{dvol}^{S^{m-1}}\ \mathrm{d}t \\ &= \int_0^r t^{m-1}\left(\operatorname{vol}(S^{m-1}) - \tfrac{t^2}{6}\int_{S^{m-1}} \operatorname{ric}(v,v)\ \mathrm{dvol}^{S^{m-1}} + \mathcal{O}(t^3)\right)\ \mathrm{d}t \\ &= \frac{r^m}{m}m\omega_m - \frac{r^{m+2}}{6(m+2)}\operatorname{scal}(p)\omega_m + \mathcal{O}(r^{m+3}) \\ &= \omega_m r^m\left(1 - \frac{r^2}{6(m+2)}\operatorname{scal}(p) + \mathcal{O}(r^3)\right).\end{aligned}$$