# Solutions 8

**1. Locally symmetric spaces**

Let $M$ be a connected $m$-dimensional Riemannian manifold. Then $M$ is called *locally symmetric* if for all $p \in M$ there is a normal neighborhood $B(p,r)$ such that the *local geodesic reflection* $\sigma_p := \exp_p \circ(-\operatorname{id}) \circ \exp_p^{-1} \colon B(p,r) \to B(p,r)$ is an isometry.

(a) Show that if $M$ is locally symmetric, then $DR \equiv 0$.

[Use that $d(\sigma_p)_p = -\operatorname{id}$ on $TM_p$.]

(b) Suppose that $DR \equiv 0$. Show that if $c \colon [-1,1] \to M$ is a geodesic and $\{E_i\}_{i=1}^m$ is a parallel orthonormal frame along $c$, then $R(E_i, c')c' = \sum_{k=1}^m r_i^k E_k$ for *constants* $r_i^k$.

(c) Show that if $DR \equiv 0$, then $M$ is locally symmetric.

[Let $q \in B(p,r), q \neq p$, and $v \in TM_q$. To show that $|d(\sigma_p)_q(v)| = |v|$, consider the geodesic $c \colon [-1,1] \to B(p,r)$ with $c(0) = p, c(1) = q$, and a Jacobi field $Y$ along $c$ with $Y(0) = 0$ and $Y(1) = v$. Use (b).]

*Solution.* (a) Suppose that $M$ is locally symmetric, let $p \in M$ and $w, x, y, z \in TM_p$. Then, since $\sigma_p$ is an isometry and $d(\sigma_p)_p = -\operatorname{id}$ on $TM_p$ we have

$$\begin{aligned} -(D_w R)(x,y)z &= d(\sigma_p)_p (D_w R)(x,y)z \\ &= (D_{d(\sigma_p)_p w})(d(\sigma_p)_p x, d(\sigma_p)_p y) d(\sigma_p)_p z \\ &= (D_{-w} R)(-x,-y) - z \\ &= (D_w R)(x,y)z, \end{aligned}$$

so $(D_w R)(x,y)z = 0$.

b) Recall that for $X, Y, Z, W \in \Gamma(TM)$

$$\begin{aligned} D_W(R(X,Y)Z) =& R(X,Y)D_W(Z) + R(D_W X, Y) \\ &+ R(X, D_W Y)Z + (D_W R)(X,Y)Z. \end{aligned}$$

Now, write $R(E_i, c')c' = \sum_{k=1}^m f_i^k E_k$ for some functions $f_i^k \colon [-1,1] \to \mathbb{R}$.

Since $E_i$ and $c'$ are parallel vector fields, the above relation implies that

$$\begin{aligned}
0 &= (D_{\partial/\partial t}R)(E_i, c')c' \\
&= D_{\partial/\partial t}\big(R(E_i, c')c'\big) \\
&= \sum_{k=1}^{m} D_{\partial/\partial t}(f_i^k E_k) \\
&= \sum_{k=1}^{m} \big(\dot{f}_i^k E_k + f_i^k D_{\partial/\partial t} E_k\big) \\
&= \sum_{k=1}^{m} \dot{f}_i^k E_k,
\end{aligned}$$

hence the $f_i^k$ are constant.

c) Let $q \in B(p, r)$, $q \neq p$ and $v \in TM_q$. We must show that $|d(\sigma_p)_q(v)| = |v|$. Let $c\colon [-1, 1] \to M$ be the geodesic with $c(0) = p$ and $c(1) = q$. Let $Y$ be the Jacobi field along $c$ with $Y(0) = 0$ and $Y(1) = v$. Since $\sigma_p$ reverts geodesics it follows that $d(\sigma_p)_q Y(1) = Y(-1)$, so it remains to show that $|Y(1)| = |Y(-1)|$. Write $Y = \sum_{i=1}^m h^i E_i$ for some functions $h^i\colon [-1, 1] \to \mathbb{R}$ then the Jacobi equation implies that

$$\ddot{h}^k + \sum_{i=1}^{m} h^i r_i^k = 0,$$

with $h^i(0) = 0$, for $k = 1, \ldots, m$. It follows that $h^i(-t) = -h^i(t)$ for all $t \in [-1, 1]$. In particular $|Y(-1)| = |Y(1)|$.

**2. Conjugate points in manifolds with curvature bounded from above**

(a) Prove directly, without using the Rauch Comparison Theorem, that there are no conjugate points in manifolds with non-positive sectional curvature.

(b) Show that in manifolds with sectional curvature at most $\kappa$, where $\kappa > 0$, there are no conjugate points along geodesics of length $< \pi/\sqrt{\kappa}$.

(c) Show that if $c\colon [0, \pi/\sqrt{\kappa}] \to M$ is a unit speed geodesic in a manifold with $\sec \geq \kappa > 0$, then some $c(t)$ is conjugate to $c(0)$ along $c|_{[0,t]}$.

*Solution.* (a) Let $Y$ be a Jacobi field along some geodesic $c\colon [0, l] \to M$ with $Y(0) = 0$ and define $f\colon [0, l] \to \mathbb{R}$, $f(t) := |Y(t)|^2 \geq 0$. By our assumption, we have $R(Y, c', Y, c') \leq 0$ and therefore

$$\begin{aligned} f'(t) &= 2\langle Y(t), Y'(t)\rangle \\ f''(t) &= 2\langle Y'(t), Y'(t)\rangle + 2\langle Y(t), Y''(t)\rangle \\ &= 2|Y'(t)|^2 - 2R(Y(t), c'(t), Y(t), c'(t)) \geq 2|Y'(t)|^2 \geq 0. \end{aligned}$$

This implies that $f$ is convex and hence, if $Y(t) = 0$ for some $t > 0$, we get $f|_{[0,t]} \equiv 0$, i.e. $Y \equiv 0$.

(b) First, consider the model space $M_\kappa$ with constant sectional curvature $\kappa$. Let $\bar{c}\colon [0, l] \to M_\kappa$ be a geodesic with $|\bar{c}'(t)| = 1$ and $\overline{Y}$ a Jacobi field along $\bar{c}$ with $\overline{Y}(0) = 0$. Such a Jacobi field is given by

$$\overline{Y}(t) = at\bar{c}'(t) + b\sin(\sqrt{\kappa}t)N(t),$$

where $N$ is a normal and parallel vector field along $\bar{c}$, compare 1 in Serie 7. In particular, we have $|\overline{Y}(t)| > 0$ for $0 < t < \pi/\sqrt{\kappa}$, $(a, b) \neq (0, 0)$ and therefore, $\bar{c}(t)$ is not conjugate to $\bar{c}(0)$ along $\bar{c}$.

For a manifold $M$ with $\sec \leq \kappa$, we can now apply the Rauch Comparison Theorem for $M$ and $M_\kappa$. We conclude that if $Y$ is a Jacobi field with $Y(0) = 0$ and $Y'(0) \neq 0$ along some geodesic $c\colon [0, l] \to M$ with $L(c) < \pi/\sqrt{\kappa}$, we have $|Y(t)| \geq |\overline{Y}(t)| > 0$.

(c) Assume that there are no conjugate points along $c$.

Let $\bar{c}\colon [0, \pi/\sqrt{\kappa}] \to M_\kappa$ be a geodesic and consider the Jacobi field $\overline{Y}(t) = \sin(\sqrt{\kappa}t)N(t)$ for some normal and parallel vector field $N$ along $\bar{c}$. Furthermore, let $Y$ be a normal Jacobi field along $c$ with $Y(0) = 0$ and $|Y'(0)| = |\overline{Y}'(0)|$. But then we get by the Rauch Comparison Theorem that $|\overline{Y}(\frac{\pi}{\sqrt{\kappa}})| \geq |Y(\frac{\pi}{\sqrt{\kappa}})| > 0$, a contradiction.

**3. Volume comparison**

Let $M$ be an $m$-dimensional Riemannian manifold with sectional curvature $\sec \leq \kappa$, $p \in M$ and $r > 0$ such that $\exp_p|_{B_r(0)}$ is a diffeomorphism. Furthermore, let $V_\kappa^m(r)$ denote the volume of a ball with radius $r$ in the $m$-dimensional model space $M_\kappa^m$ of constant sectional curvature $\kappa \in \mathbb{R}$. Prove that $V(B_r(p)) \geq V_\kappa^m$.

*Solution.* Note first that if $\kappa > 0$, then $V_\kappa^m(r) = V_\kappa^m(D_\kappa)$ for all $r > D_\kappa := \pi/\sqrt{\kappa}$ (the diameter of $M_\kappa^m$). Hence, if $\kappa > 0$, we may assume that $r \leq D_\kappa$.

Choose a base point $\overline{p}$ in $M_\kappa^m$ and a linear isometry $H\colon TM_p \to T(M_\kappa^m)_{\overline{p}}$. Since $\exp_p|_{B_r}$ is a diffeomorpism onto its image, we know from Proposition 1.21 that $B_r(p) = \exp_p(B_r)$. Define $F\colon B_r(p) \to B_r(\overline{p})$ by $F := \exp_{\overline{p}} \circ H \circ (\exp_p|_{B_r})^{-1}$. The proof of Corollary 3.19 shows that for all $x, w \in TM_p$ with $|x| < r$,

$$|d(\exp_p)_x(w)| \geq |d(\exp_{\overline{p}})_{Hx}(Hw)|.$$

Thus, for all $q \in B_r(p)$ and $v \in TM_q$,

$$|dF_q(v)| \leq |v|.$$

This implies that the volume distortion factor $J_F(q)$ of $F$ at $q$ is $\leq 1$. Hence,

$$V_\kappa^m(r) = V(B_{\overline{p}}(r)) = \int_{B_p(r)} J_F(q)\, dV(q) \leq V(B_p(r)).$$