Solutions 9

1. Poincaré models of hyperbolic space

Let us introduce the following two well-known models of the hyperbolic space:

$$\text{Unit ball } \{|z| < 1\} \subset \mathbb{R}^n \text{ equipped with metric } g_{ij} = \frac{4\delta_{ij}}{(1-|z|^2)^2}$$

and

$$\text{Half space } \{x^n > 0\} \subset \mathbb{R}^n \text{ equipped with metric } g_{ij} = \frac{\delta_{ij}}{(x^n)^2}.$$

a) Show that composing the transformations $y = x + (\frac{1}{2} - 2x^n)\boldsymbol{e}_n$ and $z = \boldsymbol{e}_n + (y - \boldsymbol{e}_n)|y - \boldsymbol{e}_n|^{-2}$ give an isometry between the two previous Riemannian manifolds

b) Show that, for the second model, circular arcs at $\{x^n = 0\}$ are geodesics.

c) Show that given any given point all geodesic rays $x(t)$, $t \geq 0$ emanating from it are minimizing up to arbitrarily large values of $t > 0$ (note that this is stronger than geodesic completeness).

d) Show that the sectional curvatures are constantly equal to -1.

Solution. a) We have

$$dz = (y - \boldsymbol{e}_n)|y - \boldsymbol{e}_n|^{-2}dy - 2|y - \boldsymbol{e}_n|^{-4}(y - \boldsymbol{e}_n) \cdot dy(y - \boldsymbol{e}_n),$$

$$|dz|^2 = |y - \boldsymbol{e}_n|^{-4}|dy|^2$$

$$1 - |z|^2 = (1 - 2y^n)|y - \boldsymbol{e}_n|^{-2}$$

Hence, using $|dy| = |dx|$ and $2y^n - 1 = -2x^n$ we obtain

$$\frac{4|dz|^2}{(1-|z|^2)^2} = \frac{4|dy|^2}{(1-2y^n)^2} = \frac{|dx|^2}{(x^n)^2}$$

b) In order to compute the geodesic equation we let $x_\varepsilon(t) := x(t) + \varepsilon\xi(t)$, where both x, ξ are function from (a, b) to $\{x^n > 0\}$, ξ vanishing at a and b. We have

$$0 = \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} L(x_\varepsilon) = \tfrac{d}{d\varepsilon}\big|_{\varepsilon=0} \int_a^b \frac{|x' + \varepsilon\xi'|}{(x^n + \varepsilon\xi^n)}dt = \int_a^b \frac{x' \cdot \xi'}{|x'|(x^n)} - \frac{|x'|}{(x^n)^2}\xi^n dt.$$

After integrating by parts and using that ξ is arbitrary we find

$$-\left(\frac{x'}{|x'|(x^n)}\right)' - \frac{|x'|}{(x^n)^2}\boldsymbol{e}_n = 0.$$

Also, $x(t)$ is parametrized by the arc length iff $\frac{|x'(t)|^2}{(x^n(t))^2} = 1$.

Hence, we obtain

$$\left(\frac{(x^\alpha)'}{(x^n)^2}\right)' = 0 \quad \text{for } \alpha = 1, 2, \dots, n-1 \qquad \left(\frac{(x^n)'}{(x^n)^2}\right)' + \frac{1}{x^n} = 0$$

Take now $x(t) = R\cos\theta(t)\boldsymbol{e}_1 + R\sin\theta(t)\boldsymbol{e}_n$, for some $R > 0$, with $\theta(t)$ satisfying $\theta' = \sin\theta$.

We have:

$$\left(\frac{(x^1)'}{(x^n)^2}\right)' = \left(\frac{-\sin\theta\theta'}{R\sin^2\theta}\right)' + \frac{1}{R\sin\theta} = \big(-1/R\big)' = 0$$

and

$$\begin{aligned}\left(\frac{(x^n)'}{(x^n)^2}\right)' + \frac{1}{x^n} &= \left(\frac{\cos\theta\theta'}{R\sin^2\theta}\right)' + \frac{1}{R\sin\theta} \\ &= \frac{(\operatorname{cotan}\theta)'}{R} + \frac{1}{R\sin\theta} = \frac{-\theta'}{R\sin^2\theta} + \frac{1}{R\sin\theta} = 0.\end{aligned}$$

Hence (using that the metric is invariant under translations and rotation in the first $n-1$ variables, we have shown that half circular arcs with centers on $\{x^n = 0\}$ are geodesics. Since for any point $p \in \{x^n > 0\}$ and for any unit vector $v \in \mathbb{S}^{n-1}$ there is a (unique) half circular arc with center on $\{x^n = 0\}$ through p and tangent to v, these are *all* geodesics.

c) The geodesic completeness follows from the fact that $\theta(t)$ above (satisfying $\theta' = \sin\theta$) is the arc length and $\int_a^b \frac{d\theta}{\sin\theta} \to +\infty$ if $a \downarrow 0$ or $b \uparrow \pi$. Also, since given any two points in$\{x^n > 0\}$ there is a unique half circular arc with center on $\{x^n = 0\}$ through them, this must be the minimizing geodesic joining them. As a consequence, any geodesic joining any two points is minimizing.

d) By Koszul's formula, for $\alpha, \beta = 1, 2, \dots, n-1$ we have

$$\nabla_{\partial_\alpha}\partial_\beta = (x^n)^{-1}\delta_{\alpha\beta}\partial_n, \quad \nabla_{\partial_n}\partial_\alpha = (x^n)^{-1}\partial_\alpha, \quad \nabla_{\partial_n}\partial_n = (x^n)^{-1}\partial_n$$

Hence,

$$\nabla_{\partial_\beta}\nabla_{\partial_\alpha}\partial_\beta = -(x^n)^2\delta_{\alpha\beta}\partial_\beta, \qquad \nabla_{\partial_\alpha}\nabla_{\partial_\beta}\partial_\beta = -(x^n)^{-2}\partial_\alpha.$$

This implies

$$R(\partial_\beta, \partial_\alpha)\partial_\beta = -(x^n)^2\partial_\alpha.$$

Similarly,

$$R(\partial_n, \partial_\alpha)\partial_n = -(x^n)^2\partial_\alpha.$$

This implies that the sectional curvatures are constantly equal to -1.

2. "Uniqueness" and symmetries of the hyperbolic space

Prove that if M is a n-dimensional Riemannian manifold satisfying properties c) and d) in the previous exercise and $p \in M$ then $\exp_p$ induces an isometry between $\mathbb{R}^n$ with metric

$$g(w,w) = \big(w \cdot \tfrac{x}{|x|}\big)^2 + \big(|w|^2 - \big(w \cdot \tfrac{x}{|x|}\big)^2\big)\frac{\sinh^2|x|}{|x|^2} \tag{1}$$

and M. Deduce that given any two points p, q in the hyperbolic space $\mathbb{H}$ and any isometry between their tangent spaces $T\mathbb{H}_p \to T\mathbb{H}_q$ there is a unique isometry $f : \mathbb{H} \to \mathbb{H}$ such that $f(p) = q$ and $df_p = H$.

Solution. Let $c(t)$ be a geodesic on M and $Y(t)$ a Jacobi field. Take $E(t)$ parallel and orthogonal to $\dot{c}(t)$. Then, since M has sectional curvatures constantly equal to -1, $Y = fE$ satisfies the Jacobi field equation provided $f'' - f = 0$, which has solutions cosh and sinh.

Notice that $t \mapsto (\exp_p)((v + \varepsilon w)t)$ gives a geodesic for all ε, for all fixed $v, w \in TM_p$. Hence, the variation $Y(t) = d(exp_p)_{vt}(wt) = td(exp_p)_{vt}(w)$ is a Jacobi field. Hence As shown in the lecture, this fact and Gauss' lemma allows us to compute $|d(\exp_p)_v(w)|$ as

$$|d(\exp_p)_v(w)|^2 = (w \cdot \tfrac{v}{v})^2 + \big(|w|^2 - \big(w \cdot \tfrac{v}{|v|}\big)^2\big)\frac{\sinh^2|v|}{|v|^2}.$$

In other words the metric of M in normal coordinates x is given by (1).

Also, since by assumption M satisfies the property c) in the previous excercise we obtain that the map $\exp_p : TM_p \to M$ is injective (and a diffeomorphism). It follows that M is isometric to $\mathbb{R}^n$ with metric g given by (1).

Finally, since we can replace p by any other point q and the expression of g in local coordinate given by $\exp_q$ will be the same, and since the metric is clearly rotationally invariant, it follows that for any isometry between $T\mathbb{H}_p \to T\mathbb{H}_q$ there is a unique isometry $f : \mathbb{H} \to \mathbb{H}$ such that $f(p) = q$ and $df_p = H$ (with is given by $(\exp_q) \circ H \circ (\exp_p)^{-1}$.

3. Translations

Suppose that Γ is a group of translations of $\mathbb{R}^m$ that acts freely and properly discontinuously on $\mathbb{R}^m$.

a) Show that there exist linearly independent vectors $v_1, \ldots, v_k \in \mathbb{R}^m$ such that

$$\Gamma = \Big\{x \mapsto x + \sum_{i=1}^{k} z_i v_i : (z_1, \ldots, z_k) \in \mathbb{Z}^k\Big\} \simeq \mathbb{Z}^k.$$

b) Let l denote the infimum of the lengths of all closed curves in $\mathbb{R}^m/\Gamma$ that are not null-homotopic. Show that l equals the length of the shortest non-zero vector of the form $\sum_{i=1}^k z_i v_i$ with $z_i \in \mathbb{Z}$ as above.

Solution. a) For each $g \in \Gamma$ there is some $v_g \in \mathbb{R}^m$ such that $gx = x + v_g$ for all $x \in \mathbb{R}^m$ and since Γ acts freely, we have $v_g \neq 0$ for $g \neq \mathrm{id}$. We denote $V := \{v_g \in \mathbb{R}^m : g \in \Gamma\}$. Note that, as Γ acts properly discontinuously, $V \cap B_r(0)$ is finite for all $r > 0$ and thus each subset of V has an element of minimal length.

We now do induction on m. For $m = 1$, choose $g \in \Gamma \setminus \{\mathrm{id}\}$ such that $|v_g|$ is of minimal length. If there is some $v \in V$ with $v = \lambda v_g$, $\lambda \notin \mathbb{Z}$, we also have $w := v - \lfloor \lambda \rfloor v_g \in V \setminus \{0\}$ with $|w| < |v_g|$, a contradiction to minimality.

For $m \geq 2$, let $v_g \in V \setminus \{0\}$ be of minimal length and let $V' := \mathrm{span}(v_g) \cap V$. By the same argument as above, we get $V' = \mathbb{Z} v_g$.

Then we have $\mathbb{R}^m = \mathbb{R}^{m-1} \oplus \mathbb{R} v_g$ with projection map $\pi\colon \mathbb{R}^m \to \mathbb{R}^{m-1}$ and $\Gamma' := \Gamma / g\mathbb{Z}$ acts by translations on $\mathbb{R}^{m-1}$ via $[h]x = x + \pi(v_h)$. As for $h \notin g\mathbb{Z}$ we have $\pi(v_h) \neq 0$, this action is free. We claim that it is properly discontinuous as well. If not, there are $(h_n)_{n \in \mathbb{N}} \in \Gamma$ with $\pi(v_{h_n}) \neq \pi(v_{h_{n'}})$ and $|\pi(v_{h_n})| < r$ for some $r > 0$. But then, there are $l_n \in \mathbb{Z}$ such that $|v_{h_n} - \pi(v_{h_n}) - l_n v_g| < |v_g|$, i.e. $(v_{h_n - l_n g})_{n \in N}$ is an infinite subset of $V \cap B_{r + |v_g|}(0)$, contradicting that Γ acts properly discontinuously.

By our induction hypothesis, there are $h_2, \ldots, h_k \in \Gamma$ such that

$$\pi(V) = \mathbb{Z}\pi(v_{h_2}) \oplus \ldots \oplus \mathbb{Z}\pi(v_{h_k})$$

and consequently $V = \mathbb{Z} v_g \oplus \mathbb{Z} v_{h_2} \oplus \ldots \oplus \mathbb{Z} v_{h_k}$.

b) Let $\pi\colon \mathbb{R}^m \to \mathbb{R}^m/\Gamma$ denote the covering map and let $c\colon [0,1] \to \mathbb{R}^m/\Gamma$ be a closed curve in $\mathbb{R}^m/\Gamma$. Then for $p \in \pi^{-1}(c(0))$, there exists a unique lift $\overline{c}\colon [0,1] \to \mathbb{R}^m$ of c with $\overline{c}(0) = p$. Furthermore, if c is not null-homotopic, we have $q := \overline{c}(1) \neq \overline{c}(0)$ and therefore

$$L(c) = L(\overline{c}) \geq d(p,q) = \left| \sum_{i=1}^k z_i v_i \right|,$$

for some $(z_1, \ldots, z_k) \in \mathbb{Z}^k \setminus \{0\}$.

Finally, if $v = \sum_{i=1}^k z_i v_i \neq 0$ is of minimal length, then $c\colon [0,1] \to \mathbb{R}^m/\Gamma$, $c(t) := \pi(tv)$, has length $L(c) = |v|$.