ETH Zürich, FS 2023
D-MATH
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Applied Stochastic Processes

Exercise sheet 1

Quiz 1.1 [**Markov chains**] A die is rolled repeatedly. The outcome is a sequence of i.i.d. (independent, identically distributed) random variables $(\xi_n)_{n\geq 1}$ with $\mathbb{P}[\xi_1 = a] = 1/6$, $\forall a \in \{1, 2, \ldots, 6\}$. We consider the following sequences of random variables $X = (X_n)_{n\geq 1}$:

i) Let X_n denote the number of rolls at time n since the most recent six.

ii) Let $X_n = \max\{\xi_1, \ldots, \xi_n\}$, the largest number that has come up in the first n rolls.

iii) Let $X_n = \max\{\xi_{n-1}, \xi_n\}$, the larger number of those that came up in the rolls number $n-1$ and n (the last two rolls). Here, we consider $(X_n)_{n\geq 2}$.

iv) Let $X_n = |\{\xi_1, \ldots, \xi_n\}|$, the number of different outcomes in the first n rolls.

For i)-iv):

(a) Determine the set of values S that the random variables can take.

(b) Decide if X is a Markov chain $\mathrm{MC}(\mu, P)$ for some initial μ and some transition probability P.

Whenever you answered (b) positively:

(c) Determine μ and P.

(d) Represent P as a weighted oriented graph.

(e) Can you determine the n-step transition probability?

Exercise 1.2 [Deterministic Markov chains]

(a) Show that a deterministic sequence $(x_n)_{n\geq 0}$ is a Markov chain if and only if there exists a function $\Phi : S \to S$ such that for all $n \geq 0$,

$$x_{n+1} = \Phi(x_n).$$

(b) Determine *all* deterministic Markov chains on $S = \{1, 2, 3\}$.

Exercise 1.3 [Matrix representation]
Consider a Markov chain $X \sim \mathrm{MC}(\delta^i, P)$ on $S = \{1, \ldots, N\}$, where $i \in S$. For $A \subseteq S$, define the $N \times N$ diagonal matrix δ_A by $(\delta_A)_{ij} = \mathbb{1}_{i=j\in A}$. Recall from Proposition 1.1 in the lecture notes that for $n \geq 0$, $j \in S$,

$$\mathbb{P}[X_n = j] = p_{ij}^{(n)} = (\delta^i P^n)_j,$$

where we identified δ_i with the row vector $(0, \ldots, 0, 1, 0, \ldots, 0)$, which has a 1 in the i^{th} position, and we wrote $P = (p_{ij})_{i,j\in S}$ for the transition probability matrix.

(a) Let $k \geq 1$, $n_k \geq \ldots \geq n_1 \geq 0$, and $i_1, \ldots, i_k \in S$. Write

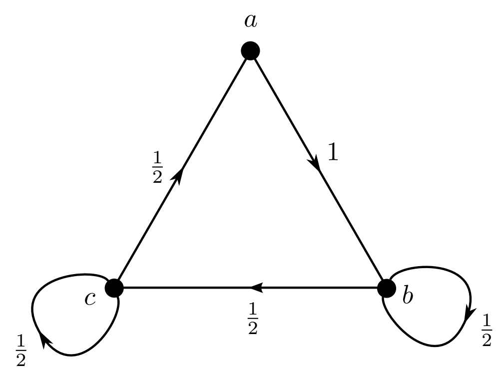
$$\mathbb{P}[X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k] = (\delta^i R)_{i_k}$$

for some matrix R and express R explicitly as a product of matrices.

(b) Let $k \geq 1$, $n_k \geq \ldots \geq n_1 \geq 0$, and $A_1, \ldots, A_k \subseteq S$. Give a similar expression for

$$\mathbb{P}[X_{n_1} \in A_1, X_{n_2} \in A_2, \ldots, X_{n_k} \in A_k].$$

Exercise 1.4 [n-step transition probability] Consider the three-state Markov chain with transition probability P given by the following diagram



Prove that

$$p_{a,a}^{(n)} = \frac{1}{5} + \left(\frac{1}{2}\right)^n \left(\frac{4}{5}\cos\frac{n\pi}{2} - \frac{2}{5}\sin\frac{n\pi}{2}\right).$$

Hint: What are the values of $p_{a,a}^{(0)}$, $p_{a,a}^{(1)}$, and $p_{a,a}^{(2)}$? Recall that $p_{a,a}^{(n)} = \delta^a P^n \delta^a$, where P is the representation of the transition probability as a 3×3 matrix. To find an expression for $\delta^a P^n \delta^a$, first compute the eigenvalues of P.

Exercise 1.5 [Construction of random variables] In this exercise, we rely on the existence of the Lebesgue measure. In particular, this provides us with the existence of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable $U : \Omega \to \mathbb{R}$ that is uniformly distributed on $[0, 1]$; denoted $U \sim \mathcal{U}([0, 1])$. Our goal is to construct random variables with other distributions as well as sequences of independent random variables on this probability space.

(a) Let $F : \mathbb{R} \to [0, 1]$ be an arbitary distribution function. Construct a random variable X with distribution function F.

Hint: Consider the generalized inverse of F, which is the mapping $F^{-1} : (0, 1) \to \mathbb{R}$ defined by $F^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}$ for $\alpha \in (0, 1)$.

(b) Construct a sequence $(Y_i)_{i \geq 1}$ of independent, Bernoulli(1/2)-distributed random variables.

Hint: Consider the binary representation of U.

(c) Construct a sequence $(U_i)_{i \geq 1}$ of independent, $\mathcal{U}([0, 1])$-distributed random variables.

(d) Let $(F_i)_{i \geq 1}$ be a sequence of arbitary distribution functions. Construct a sequence $(X_i)_{i \geq 1}$ of indenpendent random variables such that for every $i \geq 1$, X_i has distribution function F_i.

Submission deadline: 10:15, Feburary 28.

Please submit your solutions as a hard copy before the beginning of the lecture.
Further information are available on:
https://metaphor.ethz.ch/x/2023/fs/401-3602-00L/