ETH Zürich, FS 2023
D-MATH
Prof. Vincent Tassion

Ritvik Radhakrishnan
Laurin Köhler-Schindler

Applied Stochastic Processes

Exercise sheet 10

Quiz 10.1

(a) Give an example of a measure μ on $(E, \mathcal{E}) = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that is not σ-finite.

Let U be a uniformly distributed random variable taking values in $[0, 5]$.

(b) Is δ_U a point process on $([0, 5], \mathcal{B}([0, 5]))$?

(c) Is $2 \cdot \delta_U$ a point process on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$?

(d) Is δ_U a Poisson point process on $([0, 5], \mathcal{B}([0, 5]))$?

Let $(\mu_i)_{i \geq 1}$ be σ-finite measures. Let M_i, $i \geq 1$, be independent Poisson point processes on $(E, \mathcal{E})$ with respective intensity measures μ_i.

(e) Is $M_1 + M_2$ a Poisson point process on $(E, \mathcal{E})$?

(f) Is $\sum_{i=1}^{\infty} M_i$ a Poisson point process on $(E, \mathcal{E})$?

Exercise 10.2 [Poisson point process on $\mathbb{R} \times \{0, 1\}$]
Let μ be a σ-finite measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and ν be the counting measure on $(\{0, 1\}, \mathcal{B}(\{0, 1\}))$. Our goal is to construct a Poisson point process on $(E, \mathcal{E}) = (\mathbb{R} \times \{0, 1\}, \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\{0, 1\}))$ whose intensity measure is the product measure $\mu \otimes \nu$.

(a) Let μ_0 and μ_1 be the restrictions of $\mu \otimes \nu$ to $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$. Let M_0 and M_1 be two independent Poisson point process on $(E, \mathcal{E})$ with respective intensity measures μ_0 and μ_1. Show that $M_0 + M_1$ is a Poisson point process with intensity measure $\mu \otimes \nu$.

(b) Let M be a Poisson point process on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with intensity measure μ and let M' be a Poisson point process on $(\{0, 1\}, \mathcal{B}(\{0, 1\}))$ with intensity measure ν, independent of M. Define $\widetilde{M} = (\widetilde{M}(B))_{B \in \mathcal{E}}$ via

$$\widetilde{M}(A \times A') = M(A) \cdot M(A')$$

for $A \in \mathcal{B}(R)$ and $A' \in \mathcal{B}(\{0, 1\})$ and note that this definition can naturally be extended to any $B \in \mathcal{E}$. Prove or disprove that $\widetilde{M}$ is a Poisson point process on $(E, \mathcal{E})$ with intensity measure $\mu \otimes \nu$.

Exercise 10.3 [Sums of independent Poisson random variables]

(a) Consider $k \geq 1$ and $\lambda_1, \ldots, \lambda_k \in [0, \infty)$. Let $X_i \sim \text{Pois}(\lambda_i)$, $1 \leq i \leq k$, be independent random variables. Show that

$$X_1 + \ldots + X_k \sim \text{Pois}(\lambda_1 + \ldots + \lambda_k).$$

(b) Consider $(\lambda_i)_{i \geq 1}$ with $\lambda_i \in [0, \infty]$, $i \geq 1$, and set $\lambda := \sum_{i=1}^{\infty} \lambda_i$. Let $X_i \sim \text{Pois}(\lambda_i)$, $i \geq 1$, be independent random variables. Show that

$$X := \sum_{i=1}^{\infty} X_i \sim \text{Pois}(\lambda).$$

Recall the convention that $X \sim$ Pois(∞)-distributed if and only if $X = +\infty$ almost surely.

Exercise 10.4 [Campbell's formula]
Let M be a Poisson point process on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with intensity measure μ, and let $u : \mathbb{R} \to \mathbb{R}$ be a measurable function.

(a) Show that $\int u(x)M(dx)$ is a well defined random variable.

(b) Show that if we have $u \geq 0$ or $\int |u(x)|\mu(dx) < \infty$, then

$$\mathbb{E}\left[\int u(x)M(dx)\right] = \int u(x)\mu(dx).$$

Hint: First consider $u(x) = \mathbf{1}_B(x)$ for some $B \in \mathcal{B}(\mathbb{R})$. Then use measure-theoretic induction.
Remark: Along the same lines, one can prove Campbell's formula for a Polish space $(E, \mathcal{E})$.

Submission deadline: 10:15, May 9.

Please submit your solutions as a hard copy before the beginning of the lecture.
Further information are available on:
https://metaphor.ethz.ch/x/2023/fs/401-3602-00L/