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Applied Stochastic Processes

Exercise sheet 11

Quiz 11.1

(a) Let M be a Poisson point process on $\mathbb{R} \times [0, 2]$ with intensity measure $\mu = \text{Leb}$, and let $T : (x_1, x_2) \mapsto x_1$. Is $T\#N$ a Poisson point process on $\mathbb{R}$? If yes, what is its intensity measure?

(b) Let M be a Poisson point process on $\mathbb{R}^2$ with intensity measure $\mu = \text{Leb}$, and let $T : (x_1, x_2) \mapsto x_1$. Is $T\#N$ a Poisson point process on $\mathbb{R}$? If yes, what is its intensity measure?

(c) Let M be a Poisson point process on $[0, 2]^2$ with intensity measure $\mu = \text{Leb}$, and let $T : (x_1, x_2) \mapsto x_1/2$. Is $T\#N$ a Poisson point process on $[0, 1]$? If yes, what is its intensity measure?

(d) Let M be a Poisson point process on $\mathbb{R}^2$ with intensity measure $\mu = \text{Leb}$, and let $T : (x_1, x_2) \mapsto (2x_2, 2x_1)$. Is $T\#N$ a Poisson point process on $\mathbb{R}^2$? If yes, what is its intensity measure?

(e) Let M be a Poisson point process on $\mathbb{R} \times [0, \infty)$ with intensity measure $\mu = \text{Leb}$. Are the restricted process $M_{[0,1]^2}$, $M_{[0,2]^2}$, and $M_{[2,3]^2}$ Poisson point processes? If yes, what are their intensity measures? Are they independent?

Exercise 11.2 [Mapping]
Let M be a Poisson point process on $\mathbb{R}^d$ with intensity measure $\mu = \lambda \cdot \text{Leb}(\mathbb{R}^d)$, where $\lambda > 0$. Let B_r the ball of radius r around the origin.

(a) Consider the map $T : \mathbb{R}^d \to [0, \infty)$ defined by $x \mapsto ||x||_2 := \sqrt{x_1^2 + \ldots + x_d^2}$. Determine the intensity measure of the Poisson point process $T\#M$.

(b) Prove that a.s.

$$\lim_{r\to\infty} \frac{M(B_r)}{|B_r|} = \lambda$$

where $|B_r|$ is the volume of B_r.

Hint: Consider a sequence $(r_k)_{k\geq 0}$ with $|B_{r_k}| = k$ and study $M(B_{r_k} \setminus B_{r_{k-1}})$, the number of points in the annuli $(B_{r_k} \setminus B_{r_{k-1}})_{k\geq 1}$.

Exercise 11.3 [Simple Poisson point process I]
A measure ν on $(E, \mathcal{E})$ is *diffuse* if

$$\forall x \in E, \ \nu(\{x\}) = 0,$$

and we call it *simple* if

$$\forall x \in E, \ \nu(\{x\}) \leq 1.$$

Let M be a Poisson point process on $(E, \mathcal{E})$ with intensity measure μ. In this exercise, we show that

$$\mu \text{ is diffuse} \quad \Longrightarrow \quad \text{M is simple a.s..}$$

From now on, assume that μ is diffuse, and let $(E_i)_{i \in \mathbb{N}}$ be a partition of E such that each E_i is measurable and satisfies $\mu(E_i) < \infty$.

(a) Show that the restricted measures $\mu_1, \mu_2, \ldots$ are diffuse and that $M_{E_1}, M_{E_2}, \ldots$ are independent Poisson point processes with respective intensities $\mu_{E_1}, \mu_{E_2}, \ldots$.

(b) Show that for every $i \in \mathbb{N}$, M_{E_i} is almost surely simple.

Hint: Using the explicit construction of M_{E_i} from Proposition 6.10, prove that $\sum_{i=1}^{Z} \delta_{X_i}$ is almost surely simple.

(c) Deduce that M is almost surely simple.

Exercise 11.4 [Simple Poisson point process II]

(a) Let M be a Poisson point process on $\mathbb{R} \times [0, \infty)$ with intensity measure $\mu = \text{Leb}$. Is the process almost surely simple?

(b) Let M be a Poisson point process on $\mathbb{R}$ with intensity measure μ given by $\mu(B) = |B \cap \mathbb{Z}|$ for $B \in \mathcal{B}(\mathbb{R})$. Is the process almost surely simple?

Exercise 11.5 [Law of a point process]
Consider two point processes M and M' on $(E, \mathcal{E})$, and denote their laws by P_M respectively $P_{M'}$. Show that the following statements are equivalent:

(i) $P_M = P_{M'}$

(ii) For all $A \in \mathcal{B}(\mathcal{M})$,

$$\mathbb{P}[M \in A] = \mathbb{P}[M' \in A].$$

(iii) For all $k \geq 1$, $B_1, \ldots, B_k \in \mathcal{E}$, and $n_1, \ldots, n_k \in \mathbb{N}$,

$$\mathbb{P}[M(B_1) = n_1, \ldots, M(B_k) = n_k] = \mathbb{P}[M'(B_1) = n_1, \ldots, M'(B_k) = n_k].$$

(iv) For all $k \geq 1$, $C_1, \ldots, C_k \in \mathcal{E}$ disjoint, and $n_1, \ldots, n_k \in \mathbb{N}$,

$$\mathbb{P}[M(C_1) = n_1, \ldots, M(C_k) = n_k] = \mathbb{P}[M'(C_1) = n_1, \ldots, M'(C_k) = n_k].$$

Submission deadline: 10:15, May 16.

Please submit your solutions as a hard copy before the beginning of the lecture.
Further information are available on:
https://metaphor.ethz.ch/x/2023/fs/401-3602-00L/