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Applied Stochastic Processes

Exercise sheet 12

Quiz 12.1 [Poisson point processes on product space]
Let M be a Poisson point process on $(E, \mathcal{E})$ with (σ-finite) intensity measure μ and let M' be a Poisson point process on $(F, \mathcal{F})$ with (σ-finite) intensity measure ν. Define N via the product measure $N(\omega) := M(\omega) \otimes M'(\omega)$ on $(E \times F, \mathcal{E} \otimes \mathcal{F})$.

(a) Is N a point process on $(E \times F, \mathcal{E} \otimes \mathcal{F})$?

(b) Is N a Poisson point process on $(E \times F, \mathcal{E} \otimes \mathcal{F})$? If yes, what is its intensity measure?

Let M be a Poisson point process on $(E, \mathcal{E})$ with (σ-finite) intensity measure μ.

(c) Let ν be a probability measure on $(F, \mathcal{F})$. Can we construct a Poisson point process on $(E \times F, \mathcal{E} \otimes \mathcal{F})$ with intensity measure $\mu \otimes \nu$ by marking the process M?

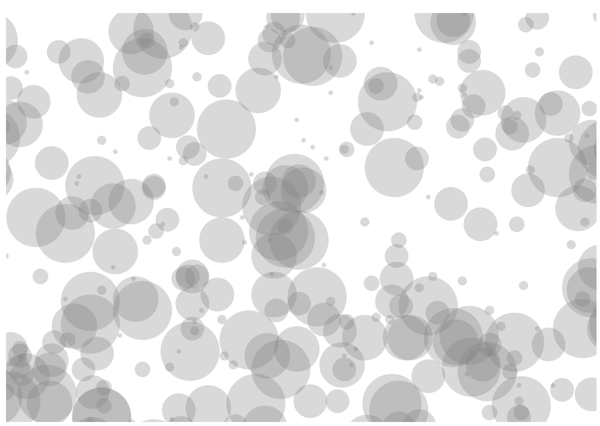
(d) Let ν be a finite measure on $(F, \mathcal{F})$. Can we use a similar construction as in (c) to construct a Poisson point process on $(E \times F, \mathcal{E} \otimes \mathcal{F})$ with intensity measure $\mu \otimes \nu$?

Exercise 12.2 [Poisson Boolean percolation]
Let $M = \sum_i \delta_{X_i}$ be a Poisson point process on $\mathbb{R}^d$ with intensity measure $\mu = \text{Leb}(\mathbb{R}^d)$. Let us consider $(R_i)_i$ a sequence of i.i.d. positive random variables with law ρ, and independent of M. We define the *occupied* set by $\mathcal{O} = \bigcup_i B(X_i, R_i)$, where $B(x, r) \subset \mathbb{R}^d$ is the open ball of center x and radius r.

(a) Let M_0 the number of balls $B(X_i, R_i)$ which contain the origin of $\mathbb{R}^d$. Show that M_0 is a well defined random variable with distribution Poisson $\left(\int_{\mathbb{R}^d} \int_{|x|}^{\infty} \rho(dr)\mu(dx)\right)$.

Hint: Use the marking theorem.

(b) Show that the event $\{\mathcal{O} = \mathbb{R}^d\}$ is measurable and that $\mathbb{P}[\mathcal{O} = \mathbb{R}^d] = 1$ if and only if $\int_0^{\infty} r^d \rho(dr) = \infty$.



Exercise 12.3 [Laplace functional]
Let M be a Poisson point process on $(E, \mathcal{E})$ with intensity measure μ. Recall that the Laplace functional $\mathcal{L}_M$ of N is given by

$$\mathcal{L}_M(u) = \mathbb{E}\left[\exp\left(-\int_E u(x)M(dx)\right)\right].$$

for every $u : E \to \mathbb{R}_+$ measurable.

(a) Let $B \in \mathcal{E}$. Show that if $\mu(B) < \infty$, then

$$\mu(B) = -\frac{d}{dt}\mathcal{L}_M(t1_B)\Big|_{t=0}.$$

(b) Let $B \in \mathcal{E}$. We no longer assume that $\mu(B) < \infty$. Show that

$$\mathbb{P}[M(B) = 0] = \lim_{t\to\infty} \mathcal{L}_M(t1_B).$$

Submission deadline: 10:15, May 23.

Please submit your solutions as a hard copy before the beginning of the lecture.
Further information are available on:
https://metaphor.ethz.ch/x/2023/fs/401-3602-00L/