ETH Zürich, FS 2023
D-MATH
Prof. Vincent Tassion

Ritvik Radhakrishnan
Laurin Köhler-Schindler

Applied Stochastic Processes

Exercise sheet 13

Quiz 13.1

(a) Consider a stochastic process $(N_t)_{t\geq 0}$ with independent, stationary increments satisfying $N_0 = 0$ and $N_t \sim \text{Pois}(t), \forall t > 0$. Is $(N_t)_{t\geq 0}$ a Poisson process with rate 1?

(b) Give an example of a counting process whose increments are stationary but *not* independent?

Let $(N_t)_{t\geq 0}$ be a counting process.

(c) Is $(N_t)_{t\geq 0}$ almost surely càdlàg? [A function is *càdlàg* if it is right-continuous with left limits.]

(d) Let $t \geq 0$. Is the number of jumps of $(N_t)_{t\geq 0}$ in the interval $[0, t]$ almost surely finite?

Exercise 13.2 [Inhomogeneous Poisson process I]
Let $\rho : [0, \infty) \to (0, \infty)$ be a continuous function satisfying $\int_0^\infty \rho(u)du = \infty$. A counting process $(N_t)_{t\geq 0}$ is called an *inhomogeneous Poisson process* with rate ρ if it has independent increments and for all $t > s > 0$,

$$N_t - N_s \sim \text{Pois}\left(\int_s^t \rho(u)du\right).$$

(a) For which choice of the function ρ do we obtain a Poisson process with rate λ?

(b) Does $(N_t)_{t\geq 0}$ have stationary increments?

Denote the jump times of $(N_t)_{t\geq 0}$ by $(S_i)_{i\geq 1}$, and consider the counting measure $M = \sum_{i\geq 1} \delta_{S_i}$. Analogously to the proof of Theorem 7.4, it is possible to prove that M is a Poisson point process on $\mathbb{R}_+$.

(c) What is the intensity measure μ_ρ of the Poisson point process M?

(d) Is S_1, the time until the first jump, independent of $S_2 - S_1$, the time between the first and the second jump?

Exercise 13.3 [Inhomogeneous Poisson process II]
Let $\rho : [0, \infty) \to (0, \infty)$ be a continuous function satisfying $\int_0^\infty \rho(u)du = \infty$.

(a) Define $R : [0, \infty) \to [0, \infty)$ by

$$R(t) = \int_0^t \rho(u)du.$$

Show that R is a continuous, increasing bijection.

(b) Let $(N_t)_{t\geq 0}$ be an inhomogeneous Poisson process with rate ρ. Show that $(\widetilde{N}_t)_{t\geq 0}$ defined by $\widetilde{N}_t := N_{R^{-1}(t)}$ is a Poisson process with rate 1.

(c) Let $(\widetilde{N}_t)_{t\geq 0}$ be a Poisson process with rate 1. Show that $(N_t)_{t\geq 0}$ defined by $N_t := \widetilde{N}_{R(t)}$ is an inhomogeneous Poisson process with rate ρ.

Exercise 13.4 [The Waiting Time Paradox]

(a) Let $(N_t)_{t\geq 0}$ be a Poisson process with rate $\lambda > 0$. Let $(S_n)_{n\geq 1}$ be the jump times of the process. For a fixed $t > 0$, let $A_t = t - S_{N_t}$ be the time passed since the most recent jump (or after 0) in the process, and let $B_t = S_{N_t+1} - t$ be the time forward to the next jump. Let $T_1 \sim \text{Exp}(\lambda)$. Show that A_t and B_t are independent, that B_t is distributed as T_1 and that A_t is distributed as $T_1 \wedge t$.

(b) Let $L_t = A_t + B_t = S_{N_t+1} - S_{N_t}$ be the length of the inter-arrival interval covering t. Show that

$$\mathbb{E}[L_t] \to 2\mathbb{E}[T_1] \quad \text{as } t \to \infty.$$

Since L_t is the time between two consecutive jumps, we might expect $\mathbb{E}[L_t] = \mathbb{E}[T_1]$. Give an intuitive resolution of the apparent paradox.

Submission deadline: 10:15, May 30.

Please submit your solutions as a hard copy before the beginning of the lecture.
Further information are available on:
https://metaphor.ethz.ch/x/2023/fs/401-3602-00L/