ETH Zürich, FS 2023
D-MATH
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Applied Stochastic Processes

Exercise sheet 14

Quiz 14.1
Let $(N_t^1)_{t\geq 0}$, $(N_t^2)_{t\geq 0}$, and $(N_t^3)_{t\geq 0}$ be independent Poisson processes with rate $\lambda > 0$.

(a) Is $(N_t)_{t\geq 0}$ defined by $N_t = N_t^1 + N_t^2 + N_t^3$ a Poisson process? If yes, what is its rate?

(b) What is the probability that the k'th jump time of $(N_t)_{t\geq 0}$ is a jump time of $(N_t^1)_{t\geq 0}$?

(c) Is $(N_t')_{t\geq 0}$ defined by $N_t' = 2 \cdot N_t^1$ a Poisson process? If yes, what is its rate?

Exercise 14.2 $^{(\star)}$[Compound Poisson process]
Let $(N_t)_{t\geq 0}$ be a Poisson process with rate $\lambda > 0$ and $(X_k)_{k\in\mathbb{N}}$ a sequence of real-valued i.i.d. random variables with common distribution μ such that $(N_t)_{t\geq 0}$ and $(X_k)_{k\in\mathbb{N}}$ are independent. Define the process $(Z_t)_{t\geq 0}$ by

$$Z_t := \sum_{k=1}^{N_t} X_k, \quad t \geq 0.$$

$(Z_t)_{t\geq 0}$ is called a *compound Poisson process* with rate λ and *jump size distribution* μ.

(a) For $t > 0$, determine the characteristic function of Z_t.

(b) Prove that $(Z_t)_{t\geq 0}$ has stationary and independent increments.

(c) Show that if $\mathbb{P}[X_i = 1] = 1 - \mathbb{P}[X_i = 0] = p$, then $(Z_t)_{t\geq 0}$ is a Poisson process with rate λp.

Exercise 14.3 [Largest gap in a Poisson process]
Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$. The largest gap up to time t is defined as

$$L_t = \max_{k\geq 1}(S_k \wedge t - S_{k-1} \wedge t).$$

(a) Let $\varepsilon > 0$. Show that there exists almost surely some $n_0 \geq 1$ such that for all $n \geq n_0$,

$$\max_{1\leq k\leq n} T_k \leq \frac{1+\varepsilon}{\lambda} \log(n/\lambda),$$

where the $T_1, \ldots, T_n$ denote the inter-arrival times of the process.

Hint: Use Borel-Cantelli's lemma.

(b) Show that there exists almost surely some $t_0 \geq 0$ such that for all $t > t_0$,

$$N_t + 1 \leq (1+\varepsilon)t\lambda.$$

(c) Conclude that almost surely

$$\limsup_{t\to\infty} \frac{L_t}{\log t} \leq \lambda^{-1}.$$

Submission deadline: No submission.

Further information are available on:
https://metaphor.ethz.ch/x/2023/fs/401-3602-00L/