ETH Zürich, FS 2023
D-MATH
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Applied Stochastic Processes

Exercise sheet 2

Quiz 2.1 [MC(μ, P) and SRW]
Let $x, y, z \in S$ and $n \geq 0$. Consider $X \sim \mathrm{MC}(\mu, P)$ and express the following probabilities in terms of μ and P:

(a) $\mathbf{P}_x[X_1 = y, X_0 = x]$ and $\mathbf{P}_x[X_1 = x, X_0 = y]$;

(b) $\mathbf{P}_x[X_{n+2} = z, X_{n+1} = y | X_n = x]$ and $\mathbf{P}_x[X_{n+2} = z, X_{n+1} = y, X_n = x]$.

Consider the SRW on $\mathbb{Z}$ with transition probability P given by $p_{ij} = \frac{1}{2} \cdot \mathbb{1}_{|i-j|=1}$. Determine the following probabilities:

(c) $\mathbf{P}_1[X_1 = 3]$, $\mathbf{P}_1[X_2 = 3]$, $\mathbf{P}_1[X_3 = 3]$, and $\mathbf{P}_1[X_4 = 3]$;

(d) $\mathbf{P}_0[X_n = 0]$.

Exercise 2.2 [Chapman-Kolmogorov]
Consider the SRW on $\mathbb{Z}$. Recall that the transition probability P is given by $p_{ij} = \frac{1}{2} \cdot \mathbb{1}_{|i-j|=1}$.

(a) Show that for every $x \in \mathbb{Z}$ and for every $n \geq 0$,

$$p_{0x}^{(2n)} \leq \sqrt{\sum_{y \in S} \left(p_{0y}^{(n)}\right)^2} \cdot \sqrt{\sum_{y \in S} \left(p_{yx}^{(n)}\right)^2}.$$

(b) Deduce that for every $x \in \mathbb{Z}$ and for every $n \geq 0$,

$$p_{0x}^{(2n)} \leq p_{00}^{(2n)}.$$

Exercise 2.3 [Simple Markov property I]
Consider the SRW on $\mathbb{Z}$. Show that the two random variables

$$Z := \sum_{n=0}^{10} \mathbb{1}_{X_n = 0} \quad \text{and} \quad Z' := \sum_{n=10}^{20} \mathbb{1}_{X_n = X_{10}}$$

have the same distribution and are independent under $\mathbf{P}_0$.

Exercise 2.4 [Simple Markov property II]
Consider the SRW on $\mathbb{Z}$. For $N \geq 0$, we define the hitting time $H_{-N,N} := \inf\{n \geq 0 : X_n \in \{-N, N\}\}$.

(a) Show that for every $k \geq 0$,

$$\mathbf{P}_0[H_{-N,N} > k \cdot N] \leq \left(1 - 2^{-N}\right)^k.$$

Deduce that $\mathbf{E}_0[H_{-N,N}] \leq N \cdot 2^N$.

(b) Show that $\mathbf{E}_x[H_{-N,N}] < \infty$ for all $x \in \{-N, \ldots, N\}$.

(c) Prove that $\mathbf{E}_0[H_{-N,N}] = N^2$.
Hint: Consider the function $f(x) = \mathbf{E}_x[H_{-N,N}]$ for $x \in \{-N, \ldots, N\}$.

Exercise 2.5 [Complements I]

(a) [**Existence theorem**] Let S be finite or countable. Show that there exists a distribution μ on S with $\mu(x) > 0$ for every $x \in S$.

(b) [**Proposition 1.2**] Let $(X_n)_{n\geq 0}$ be a sequence of random variables with values in S satisfying the 1-step Markov property and homogeneity. Show that there exist a distribution μ and a transition probability P such that

$$X \sim \mathrm{MC}(\mu, P).$$

We note that (b) establishes the converse of Proposition 1.2, thereby showing that the 1-step Markov property and homogeneity characterize Markov chains.

Exercise 2.6 [Complements II: Strong Markov property]

(a) Show that Z is $\mathcal{F}_T$-measurable if and only if $Z \cdot \mathbb{1}_{T=n}$ is $\mathcal{F}_n$-measurable for every $n \in \mathbb{N}$.

Let μ be a probability measure on S, let T be an $(\mathcal{F}_n)_{n\geq 0}$-stopping time and let $x \in S$. For $f : S^{\mathbb{N}} \to \mathbb{R}$ measurable bounded and Z $\mathcal{F}_T$-measurable bounded, the strong Markov property gives

$$\mathbf{E}_\mu \left[f\left((X_{T+n})_{n\geq 0}\right) \cdot Z | T < \infty, X_T = x \right] = \mathbf{E}_x \left[f\left((X_n)_{n\geq 0}\right) \right] \cdot \mathbf{E}_\mu \left[Z | T < \infty, X_T = x \right].$$

(b) Assume that $\mathbf{P}_\mu[T < \infty] = 1$. Show that

$$\mathbf{E}_\mu \left[f\left((X_{T+n})_{n\geq 0}\right) \cdot Z | X_T = x \right] = \mathbf{E}_x \left[f\left((X_n)_{n\geq 0}\right) \right] \cdot \mathbf{E}_\mu \left[Z | X_T = x \right].$$

Exercise 2.7 [Exponential tail of exit time from a finite set]
Comment: This exercise is a generalization of the result in Exercise 2.4 (a).
Let $(X_n)_{n\geq 0}$ be a Markov chain with transition probabilities $(p_{x,y})_{x,y\in S}$. Let $C \subseteq S$ such that $S \setminus C$ is finite. Define $n(x) := \min\{n \geq 0 : \mathrm{P}_x[X_n \in C] > 0\}$, and suppose that $n(x) < \infty$ for all $x \in S$. Let

$$\begin{aligned} \tau_C &= \inf\{n \geq 0 : X_n \in C\}, \\ \varepsilon &= \min\{\mathrm{P}_x[X_{n(x)} \in C] : x \in S\}, \\ N &= \max\{n(x) : x \in S\}. \end{aligned}$$

Show that for all $k \in \mathbb{N}$ and for every $x \in S$,

$$\mathrm{P}_x[\tau_C > kN] \leq (1 - \varepsilon)^k.$$

Submission deadline: 10:15, March 7.

Please submit your solutions as a hard copy before the beginning of the lecture.
Further information are available on:
https://metaphor.ethz.ch/x/2023/fs/401-3602-00L/