ETH Zürich, FS 2023
D-MATH
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Applied Stochastic Processes

Exercise sheet 3

Quiz 3.1 [Quiz]
Let $x, y \in S$. For $x \in S$, we recall the definitions $H_x = \min\{n \geq 1 : X_n = x\}$,

$$V_x = \sum_{k \geq 1} \mathbf{1}_{X_k = x}, \quad \text{and} \quad V_x^{(n)} = \sum_{k=1}^{n} \mathbf{1}_{X_k = x}.$$

In addition, we define $\widetilde{H_x} = \min\{n \geq 0 : X_n = x\}$, and

$$\widetilde{V_x} = \sum_{k \geq 0} \mathbf{1}_{X_k = x}.$$

(a) What can you say about $\mathbf{P}_y[\widetilde{H_x} < \infty]$ in comparison with $\mathbf{P}_y[H_x < \infty]$?

(b) What can you say about $\mathbf{E}_y[\widetilde{V_x}]$ in comparison with $\mathbf{E}_y[V_x]$?

Let $y \neq x$.

(c) Why is it true that

$$\mathbf{E}_y[V_x] = \mathbf{P}_y[H_x < \infty] \cdot (1 + \mathbf{E}_x[V_x]) \text{ ?} \tag{1}$$

(d) For $n \geq 1$, compare $\mathbf{E}_y[V_x^{(n)}]$ and $\mathbf{E}_y[V_x^{(n+1)}]$.

(e) What is

$$\lim_{n \to \infty} \mathbf{E}_y[V_x^{(n)}]?$$

(f) Does (1) hold true when we replace V_x by $V_x^{(n)}$ on both sides of the equation?

(g) Is it possible that $\mathbf{P}_x[V_x = \infty] = 1/2$?

(h) Is it possible that $\mathbf{P}_x[V_x = 2] = 1/8$?

(i) Is it possible that $\mathbf{P}_x[V_x = 2] = 1$?

Recall the definition of the sequence $(T_i)_{i \geq 1}$ of inter-visit times at x.

(j) Under $\mathbf{P}_x$, what is

$$\lim_{n \to \infty} \frac{T_5 + \ldots + T_n}{n} \text{ ?}$$

(k) What is

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} p_{xx}^{(k)} \text{ ?}$$

Exercise 3.2 [Biased and reflected random walk]
Let $\alpha \in (0,1)$. We consider the biased random walk X on $\mathbb{Z}$, i.e. the Markov chain with state space $\mathbb{Z}$ and transition probability given by

$$p_{x,x+1} = \alpha, \quad \text{and} \quad p_{x,x-1} = 1-\alpha \quad \text{for } x \in \mathbb{Z}.$$

(a) Let $x \in \mathbb{Z}$. Show that x is recurrent if $\alpha = 1/2$.

Hint: Deduce from Exercise 2.2 (b) that $\sum_{n\geq 0} p_{xx}^{(n)} = \infty$, and apply the Dichotomy Theorem.

(b) Let $x \in \mathbb{Z}$. Show that x is transient if $\alpha \neq 1/2$.

Hint: Use the strong law of large numbers.

We now consider the reflected random walk Y on $\mathbb{N}$, i.e. the Markov chain with state space $\mathbb{N}$ and transition probability given by $p_{0,1} = 1$,

$$p_{x,x+1} = \alpha, \quad \text{and} \quad p_{x,x-1} = 1-\alpha \quad \text{for } x \geq 1.$$

(c) Show that 0 is recurrent if $\alpha \leq 1/2$.

Hint: Construct a coupling with the biased random walk X.

(d) Show that 0 is positive recurrent if $\alpha < 1/2$.

(e) Show that 0 is transient if $\alpha > 1/2$.

As a direct consequence of the Classification of States Theorem (next week), it will follow that any state x is recurrent if and only if $0 < \alpha \leq 1/2$.

Exercise 3.3 [SRW on a d-regular tree]
Let $d \geq 3$. In this exercise, we consider the simple random walk X on the d-regular tree $\mathbf{T}_d = (V, E)$, i.e. the infinite tree with d edges at each vertex. This is the Markov chain with state space V and transition probability given by

$$p_{xy} = \frac{1}{d} \cdot \mathbf{1}_{\{x,y\}\in E} \ .$$

Show that every state $x \in V$ is transient.
Hint: Under $\mathbf{P}_x$, consider the distance of X_n from the starting point x.

Exercise 3.4 [Reflection principle for the SRW on $\mathbb{Z}$]
Consider the SRW on $\mathbb{Z}$, i.e. the Markov chain with transition probability given by $p_{ij} = \frac{1}{2} \cdot \mathbb{1}_{|i-j|=1}$. The goal of this exercise is to prove that for $n \geq 0$ even and $a \geq 1$ odd,

$$\mathbf{P}_0\left[\left(\max_{0\leq m\leq n} X_m\right) \geq a\right] = \mathbf{P}_0\big[|X_n| \geq a\big]. \tag{2}$$

(a) Show that

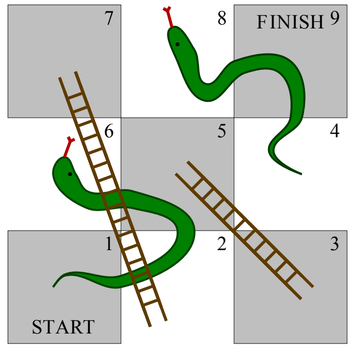
$$\mathbf{P}_0\left[\left(\max_{0\leq m\leq n} X_m\right) \geq a\right] = \mathbf{P}_0[X_n > a] + \mathbf{P}_0[H_a \leq n, X_n < a].$$

(b) Use the strong Markov property to show that

$$\mathbf{P}_0[H_a \leq n, X_n < a] = \mathbf{P}_0[X_n > a],$$

and conclude that (2) holds.

Exercise 3.5 [Snakes and ladders]
A simple game of ‘snakes and ladders’ is played on a board of nine squares.



At each turn a player tosses a fair coin and advances one or two places according to whether the coin lands heads or tails. If you land at the foot of a ladder you climb to the top, but if you land at the head of a snake you slide down to the tail.

(a) How many turns on average does it take to complete the game?
Hint: Call $k_i = \mathbf{E}_i[H_9]$ and find some relations between the k_i for $i \in \{1, \dots, 9\}$.

(b) What is the probability that a player who has reached the middle square will complete the game without slipping back to square 1?

Submission deadline: 10:15, March 14.

Please submit your solutions as a hard copy before the beginning of the lecture.
Further information are available on:
https://metaphor.ethz.ch/x/2023/fs/401-3602-00L/ document