ETH Zürich, FS 2023
D-MATH
Prof. Vincent Tassion

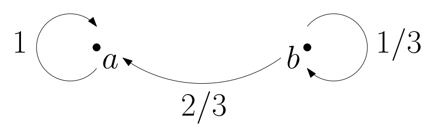
Ritvik Radhakrishnan
Laurin Köhler-Schindler

Applied Stochastic Processes

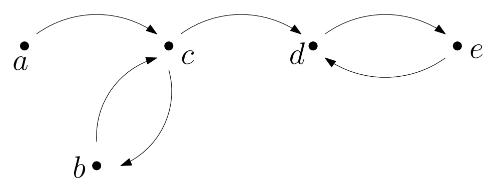
Exercise sheet 4

Quiz 4.1

(a) Consider the transition probability represented by the diagram below. Which states are recurrent? Which states are transient?



(b) Consider the transition probability represented by the diagram below. How many communication classes are there?

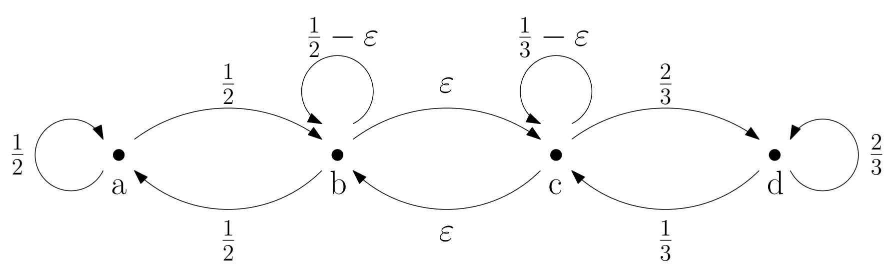


(c) Does there exist an *irreducible* Markov chain with infinitely many recurrent classes?

(d) Does there exist a Markov chain with infinitely many recurrent classes?

Quiz 4.2

For $0 \leq \varepsilon \leq \frac{1}{3}$, consider the Markov chain on $S = \{a, b, c, d\}$ with transition probability given by the following diagram.



Which of the following statements are correct (possibly depending on the value of ε)?

(a) $a \leftrightarrow b$

(b) $b \leftrightarrow d$

(c) The Markov chain is irreducible.

Exercise 4.3 [Galton-Watson chain]
Let ν be a probability measure on $\mathbb{N}$ and let $(Z_i^n)_{i,n\geq 1}$ be independent ν-distributed random variables with $\nu(0), \nu(1) > 0$ and $\nu(0) + \nu(1) < 1$. The Galton-Watson chain $(X_n)_{n\geq 0}$ is defined by $X_0 = 1$ and for $n \geq 0$ by

$$X_{n+1} = \begin{cases} Z_1^{n+1} + Z_2^{n+1} + \ldots + Z_{X_n}^{n+1} & \text{if } X_n > 0, \\ 0 & \text{if } X_n = 0. \end{cases}$$

(a) Show that $(X_n)_{n\geq 0}$ is a Markov chain $\mathrm{MC}(\delta^1, P)$ on $\mathbb{N}$, and express P in terms of ν.

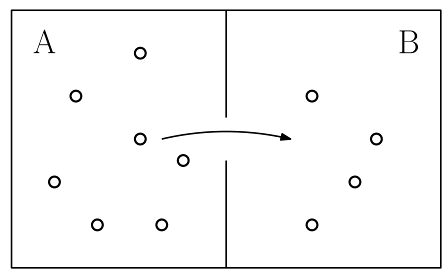
(b) Identify all communication classes. Which communication classes are closed?

(c) Which communication classes are recurrent? Which are transient?

(d) How do your answers to (b) and (c) change if $\nu(0) = 0$ and $\nu(1) < 1$?

Exercise 4.4 [Gas in Containers (Ehrenfest Model)]

Idea: Imagine two containers A and B with gas particles, and a small hole between them through which the particles can pass. At every step, a single particle is selected uniformly at random and passes through this hole.
Mathematical model: Let N be the total number of gas particles, and let X_n be the number of particles in A at time n. We want to model $(X_n)_{n\geq 0}$ as a Markov chain.



(a) Determine a suitable state space S and transition probability P.

(b) Identify a stationary distribution π for P.

Hint: Try to find a reversible distribution for P.

Exercise 4.5 [Stationary measures]
Consider a Markov chain $(X_n)_{n\geq 0}$ with state space S (finite or countable). Let $x \in S$ be recurrent and define for $y \in S$,

$$\mu_x(y) := \mathbf{E}_x\left[\sum_{n=0}^{H_x-1} \mathbf{1}_{X_n=y}\right].$$

(a) Show that μ_x is a *stationary measure*, i.e. for every $y \in S$,

$$\mu_x(y) = \sum_{z\in S} \mu_x(z) p_{zy} \quad \text{and} \quad 0 \leq \mu_x(y) < \infty.$$

What would go wrong for $x \in S$ transient?

Hint: Note that $\mathbf{E}_x\left[\sum_{n=0}^{H_x-1} \mathbf{1}_{X_n=y}\right] = \sum_{n\geq 0} \mathbf{P}_x[X_n = y, H_x > n]$.

(b) Show that for any $y \in S$,

$$\mu_x(y) = \begin{cases} \frac{\mathbf{P}_x[H_y < H_x]}{\mathbf{P}_y[H_x < H_y]} & \text{if } x \to y, \\ 0 & \text{otherwise.} \end{cases}$$

Submission deadline: 10:15, March 21.

Please submit your solutions as a hard copy before the beginning of the lecture.
Further information are available on:
https://metaphor.ethz.ch/x/2023/fs/401-3602-00L/ document