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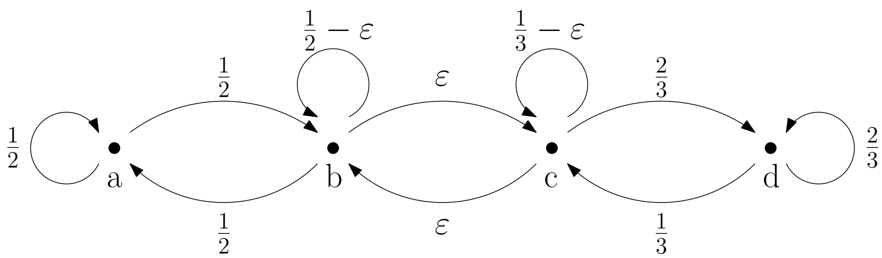
Applied Stochastic Processes

Exercise sheet 5

Quiz 5.1 [Continuation of Quiz 4.2]
For $0 \leq \varepsilon \leq \frac{1}{3}$, consider the Markov chain on $S = \{a, b, c, d\}$ with transition probability given by the following diagram.



Which of the following statements are correct (possibly depending on the value of ε)?

(a) $\pi = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ is reversible for the Markov chain.

(b) $\pi = (\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2})$ is reversible for the Markov chain.

(c) $\pi = (\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2})$ is stationary for the Markov chain.

(d) $\pi = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})$ is stationary for the Markov chain.

Quiz 5.2
Let S be finite or countable. Let P be irreducible.

(a) Let $x \in S$. Is it possible that

$$\sum_{y \in S} \frac{1}{\mathbf{E}_x[H_y]} = 2\,?$$

(b) Is it possible that

$$\sum_{y \in S} \frac{1}{\mathbf{E}_y[H_y]} = 2\,?$$

(c) Assume S finite. Does there always exist a unique stationary distribution?

Let $(X_n)_{n \geq 0}$ be i.i.d. random variables with values in S satisfying $\mathbf{P}[X_0 = x] > 0$ for all $x \in S$.

(d) Show that the Markov chain $(X_n)_{n \geq 0}$ is irreducible and positive recurrent.

Exercise 5.3 [Two-state Markov chain]
Let $p, q \in [0, 1]$. Consider the two state Markov chain on $S = \{1, 2\}$ with transition probability P given by

$$p_{11} = p, \quad p_{12} = 1 - p, \quad p_{22} = q, \quad \text{and} \quad p_{21} = 1 - q.$$

Prove that a distribution π is stationary for P if and only if it is reversible for P.

Exercise 5.4 [Biased random walk on $\{1,\dots,N\}$]
Let $\alpha \in [0,1]$. Consider the biased random walk on $S = \{1,\dots,N\}$ with transition probability given by

$$p_{ij} = \begin{cases} \alpha & \text{if } j = i+1 \,(\operatorname{mod} N), \\ 1-\alpha & \text{if } j = i-1 \,(\operatorname{mod} N). \end{cases}$$

(a) Show that π, defined by $\pi(i) = 1/N, \forall i \in \{1,\dots,N\}$, is the unique stationary distribution for the biased random walk.

(b) Show that the biased random walk is reversible if and only if $\alpha = 1/2$.

Exercise 5.5 [Lazy Markov chain]
Let $\delta \in (0,1)$, and let P be an irreducible transition probability. Define for all $x, y \in S$,

$$\widetilde{p}_{xy} = \delta \cdot \mathbb{1}_{x=y} + (1-\delta) \cdot p_{xy}.$$

(a) Show that $\widetilde{P} = (\widetilde{p}_{xy})_{x,y\in S}$ is a transition probability.

(b) Prove that $\widetilde{P}$ is irreducible and aperiodic.

(c) Assume that P positive recurrent. Prove that $\widetilde{P}$ is positive recurrent. What is the stationary distribution for $\widetilde{P}$?

Exercise 5.6 [Time-reversed Markov chain]
Let S be finite or countable. Consider a transition probability $P = (p_{xy})_{x,y\in S}$ and a stationary distribution π.

(a) Define $\hat{P} = (\hat{p}_{xy})_{x,y\in S}$ by

$$\hat{p}_{xy} = \begin{cases} \frac{\pi_y p_{yx}}{\pi_x} & \text{if } \pi_x > 0, \\ \mathbb{1}_{x=y} & \text{if } \pi_x = 0. \end{cases}$$

Show that $\hat{P}$ is a transition probability. $\mathrm{MC}(\pi, \hat{P})$ is called the *time-reversal* of $\mathrm{MC}(\pi, P)$.

(b) Recall that a transition probability P can be represented by a linear operator $P : \mathrm{L}^\infty(S) \to \mathrm{L}^\infty(S)$, defined by

$$(Pf)(x) = \sum_{y\in S} p_{xy} f(y).$$

Define an inner product $\langle \cdot, \cdot \rangle_\pi$ on $\mathrm{L}^\infty(S)$ by

$$\langle f, g \rangle_\pi := \sum_{x\in S} f(x) g(x) \pi_x.$$

Show that the operators P and $\hat{P}$ are adjoint, i.e. $\langle Pf, g\rangle_\pi = \langle f, \hat{P} g\rangle_\pi$ for every $f, g \in \mathrm{L}^\infty(S)$.

(c) Show that the operator P is self-adjoint if and only if π is reversible (for P).

Submission deadline: 10:15, March 28.

Please submit your solutions as a hard copy before the beginning of the lecture.
Further information are available on:
https://metaphor.ethz.ch/x/2023/fs/401-3602-00L/ document