ETH Zürich, FS 2023
D-MATH
Prof. Vincent Tassion

Ritvik Radhakrishnan
Laurin Köhler-Schindler

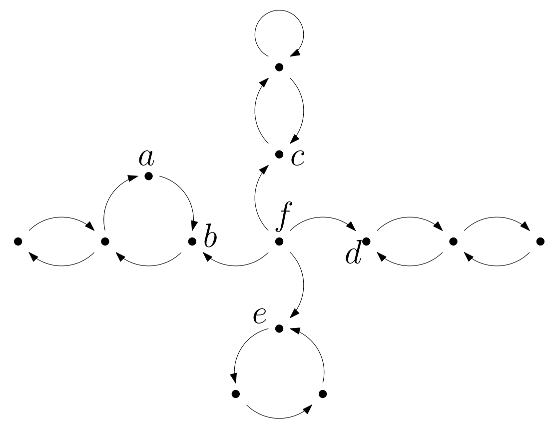
Applied Stochastic Processes

Exercise sheet 6

Quiz 6.1

(a) Give an example of an irreducible transition probability P with period 5.

(b) Consider the transition pobability represented by the following directed graph. For each $x \in \{a, b, c, d, e, f\}$, what is the period of x?

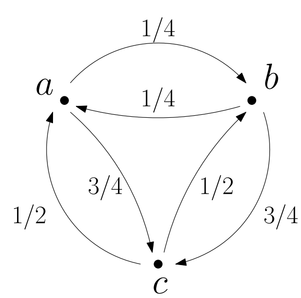


(c) Consider the biased random walk $X = (X_n)_{n\geq 0}$ on $\mathbb{Z}$ with parameter $\alpha \in (0,1)$ (see Exercise 3.2 for the definition of P). What is

$$\lim_{n\to\infty} \mathbf{P}_0[X_n = 0] \text{ ?}$$

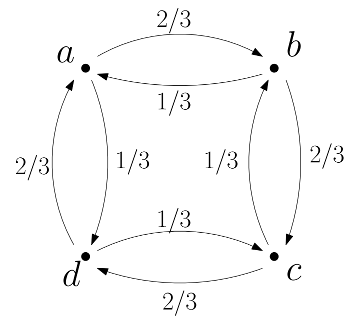
Exercise 6.2 [Convergence to equilibrium I]
Consider the three-state Markov chain X represented by the following directed graph. What is

$$\lim_{n\to\infty} \mathbf{P}_b[X_n = b] \text{ ?}$$



Exercise 6.3 [Convergence to equilibrium II]
Consider the four-state Markov chain $X = (X_n)_{n\geq 0}$ represented by the following directed graph. What is

$$\lim_{n\to\infty} \mathbf{P}_c[X_{2n} = a] \ ?$$



Hint: Determine the transition probability of the Markov chain $(X_{2n})_{n\geq 0}$ with $X_0 = c$.

Exercise 6.4 [Hardcore model I]
Recall the hardcore model from Section 3.9. In this exercise, we consider the hardcore model on a 2x2 square grid. Denoting the vertices by $V = \{a, b, c, d\}$, this is the graph $G = (V, E)$ with edge set

$$E = \Big\{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\Big\}.$$

(a) Identify all admissible configurations ξ. How many are there?

(b) Determine the transition probability P and represent it as a directed graph.

(c) Deduce that P is aperiodic and irreducible.

Exercise 6.5 [Hardcore model II]
Recall the hardcore model from Section 3.9. This is the Markov chain $X = (X_n)_{n\geq 0}$ on

$$S = \{\xi \in \{0, 1\}^V : \xi \text{ is admissible}\}$$

with transition probability P as defined in the proof of Proposition 3.17.

(a) Show that P is aperiodic.

(b) Show that P is irreducible.

In Section 3.9, this Markov chain was used to simulate a uniform random variable in S. Now, we fix $0 \leq k \leq 32$, and consider the set of admissible configurations with exactly k particles

$$S_k := \Big\{\xi \in S : \sum_{v\in V} \xi(v) = k\Big\}.$$

(c) How could we simulate Z, a uniform random variable in S_k?

Submission deadline: 10:15, April 4.

Please submit your solutions as a hard copy before the beginning of the lecture.
Further information are available on:
https://metaphor.ethz.ch/x/2023/fs/401-3602-00L/ document