ETH Zürich, FS 2023
D-MATH
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Applied Stochastic Processes

Exercise sheet 7

Quiz 7.1
Let $(N_t)_{t\geq 0}$ be a renewal process with arrival distribution F and denote $\mu := \mathbb{E}[T_1] \in (0, \infty]$.

(a) Does N_t/t converge a.s. as $t \to \infty$? If yes, what is the limit?

(b) Does N_t/t^2 converge a.s. as $t \to \infty$? If yes, what is the limit?

(c) If $\mu < \infty$, does $N_t/\sqrt{t}$ converge a.s. as $t \to \infty$? If yes, what is the limit?

(d) If $\mu = \infty$, does $N_t/\sqrt{t}$ converge a.s. as $t \to \infty$? If yes, what is the limit?

Exercise 7.2 [Renewal function]
For $(N_t)_{t\geq 0}$, a renewal process with arrival distribution F, we define the *renewal function* $m : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$t \mapsto m(t) := \mathbb{E}[N_t].$$

Draw m for the following distributions of the inter-arrival times:

(a) $T_1 = 1$ a.s.

(b) $T_1 \sim \mathcal{U}(0, 1)$

(c) $T_1 \sim \text{Exp}(2)$

(d) $T_1 \sim \text{Ber}(1/2)$

Exercise 7.3 [CLT for renewal processes]
Let $(N_t)_{t\geq 0}$ be a renewal process with arrival distribution F and denote $\mu = E[T_1]$. Assume that $\mathbb{E}[{T_1}^2] < \infty$ and $\sigma^2 := \text{Var}(T_1) > 0$. Show that

$$\frac{N_t - t/\mu}{\sigma\sqrt{t/\mu^3}} \xrightarrow{\text{law}} \mathcal{N}(0, 1) \quad \text{as } t \to \infty,$$

where $\mathcal{N}(0, 1)$ is the standard normal distribution.
Hint: Let $S_n := T_1 + ... + T_n$, then by the central limit theorem

$$\lim_{n\to\infty} P[(S_n - n\mu)/\sigma\sqrt{n} \leq x] = \Phi(x)$$

uniformly in $x \in \mathbb{R}$, where Φ denotes the distribution function of the standard normal distribution.

Exercise 7.4 [Renewal process with lattice arrival distribution]
Let $(N_t)_{t\geq 0}$ be a renewal process with arrival distribution given by T_1, and assume that the law of T_1 is lattice with span a. Denote $\mu := \mathbb{E}[T_1]$.

(a) Show that the span
$$a := \max\{a' > 0 : \mathbb{P}[T_1 \in a'\mathbb{Z}] = 1\}$$
is well-defined and finite.

(b) Show that $(\widetilde{N}_t)_{t\geq 0}$, defined by $\widetilde{N}_t := N_{at}$, is a renewal process with integer-valued jump times.

From now on, we assume that T_1 is lattice with span 1 and $\mathbb{P}[T_1 = 0] = 0$. We define

$$S := \begin{cases} \{0, 1, \ldots, N-1\} & \text{if } N := \sup\{n \geq 0 : \mathbb{P}[T_1 = n] > 0\} < \infty, \\ \mathbb{N} & \text{otherwise,} \end{cases}$$

and for $i \geq 1$,
$$p_{0,i-1} = \mathbb{P}[T_1 = i], \quad \text{and} \quad p_{i,i-1} = 1.$$

(c) Show that $p = (p_{ij})_{i,j\in S}$ is a transition probability, and that the chain P is aperiodic, irreducible, and recurrent.

As in Exercise 7.2, we define the *renewal function* $m : \mathbb{R}_+ \to \mathbb{R}_+$ by
$$t \mapsto m(t) := \mathbb{E}[N_t].$$

(d) *[Elementary renewal theorem]:* Show that
$$\lim_{t\to\infty} \frac{m(t)}{t} = \frac{1}{\mu}.$$

Hint: Use the theorem “density of visit times” for Markov chains from Section 2.6.

(e) *[Blackwell’s renewal theorem]:* Show that for all $k \in \mathbb{N}$,
$$\lim_{t\to\infty} m(t+k) - m(t) = \frac{k}{\mu}.$$

Hint: Use the results on the convergence of Markov chains from Sections 3.7 and 3.8.

Submission deadline: 10:15, April 18.

Please submit your solutions as a hard copy before the beginning of the lecture.
Further information are available on:
https://metaphor.ethz.ch/x/2023/fs/401-3602-00L/ document